Pensions Institute

DISCUSSION PAPER PI-0609

Mortality – Dependent Financial Risk Measures

Kevin Dowd, Andrew J.G. Cairns and David Blake

May 2006

ISSN 1367-580X

The Pensions Institute
Cass Business School
City University
106 Bunhill Row London
EC1Y 8TZ
UNITED KINGDOM

http://www.pensions-institute.org/

Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
Insurance: Mathematics and Economics 38 (2006) 427–440
INSURANCE MATHEMATICS & ECONOMICS
www.elsevier.com/locate/ime

Mortality-dependent financial risk measures

Kevin Dowd[a,*], Andrew J.G. Cairns[b], David Blake[c]

[a] *Centre for Risk and Insurance Studies, Nottingham University Business School, Jubilee Campus, Nottingham NG8 1BB, UK*
[b] *Actuarial Mathematics and Statistics, School of Mathematical and Computer Sciences, Heriot-Watt University, Edinburgh EH14 4AS, UK*
[c] *Pensions Institute, Cass Business School, 106 Bunhill Row, London EC1Y 8TZ, UK*

Received June 2005; received in revised form October 2005; accepted 17 November 2005

Abstract

This paper uses a recently developed two-factor stochastic mortality model to estimate financial risk measures for four illustrative types of mortality-dependent financial position: investments in zero-coupon longevity bonds; investments in longevity bonds that pay annual survivor-dependent coupons; and two examples of an insurer's annuity book that are each hedged by a longevity bond, one based on the annuity book and hedge having the same reference cohort, and the other not. The risk measures estimated are the value-at-risk, the expected shortfall and a spectral risk measure based on an exponential risk-aversion function. Results are reported on a model calibrated on data provided by the UK Government Actuary's Department, both with and without underlying parameter uncertainty.

Keywords: Mortality risk; Longevity bonds; Value-at-risk; Coherent risk measures; Spectral risk measures

1. Introduction

There has been considerable interest in recent years in models of mortality. Much of this interest has arisen from the belated recognition – evident, for example, in the Equitable Life fiasco[1] – that actuaries in the past have tended to pay insufficient attention to aggregate mortality risk. It is also becoming clear that many insurance companies have considerable exposure to this risk, and that they currently lack the tools to price and manage this risk as effectively as they should. It is therefore not surprising that a number of mortality risk models have been proposed in the past few years,[2] and that there have also been proposals for mortality derivatives, most particularly for longevity bonds (LBs).[3] The first publicly offered mortality derivative – the Swiss Re LB – was then issued in December 2003,[4] and this was followed by the European

* Corresponding author. Tel.: +44 115 846 6682; fax: +44 115 846 6667.

E-mail address: Kevin.Dowd@nottingham.ac.uk (K. Dowd).

[1] In this case, the firm offered guaranteed annuity options based on 1950s mortality tables, and its failure to take proper account of mortality risk was a key factor contributing to its being forced to close to new business in 2000.

[2] These models include those of Milevsky and Promislow (2001), Yang (2001), Cairns et al. (2006a), Dahl (2004) and Lin and Cox (2005b).

[3] Longevity bonds were first proposed by Blake and Burrows (2001) (under the name of survivor bonds), and further developed by Lin and Cox (2005a) among others. Mortality swaps were suggested by Lin and Cox (2004) and Dowd et al. (2006).

[4] This was an otherwise conventional bond whose principal payment was linked to adverse mortality risk scenarios. The face value was $400 million and the maturity of the issue was 4 years. Investors received a floating coupon rate of US LIBOR plus 135 basis points, but the principal payment was at risk if the weighted average of general population mortality across five reference countries (US, UK, France, Italy and Switzerland) should exceed 130% of the 2002 level. Since mortality is improving, the chances of such high mortality were judged to be very

doi:10.1016/j.insmatheco.2005.11.003

Investment Bank/Bank National de Paris longevity bond announced in November 2004.[5] Financial institutions have also started to trade mortality derivatives over-the-counter, and the indications are that a wholesale market in aggregate mortality risk is beginning to take shape.

These developments are encouraging, but insurance companies still have a problem: how do they assess the magnitude of the financial risks implied by their mortality exposures? This paper seeks to provide an answer to this problem. The answer suggested has three key ingredients: a stochastic model of aggregate mortality; some financial risk measures; and a simulation framework that enables us to estimate these financial risk measures for a mortality-dependent portfolio based on the underlying mortality model. In addition, since the parameters of the mortality model are (inevitably) uncertain, our estimation framework should also take account of the uncertainty attached to estimates of the parameters (i.e., parameter risk).

This paper is organized as follows. Section 2 summarizes the mortality model used in the study: it covers the main features of the model and addresses the associated issues of calibration and simulation. Section 3 looks at applications of this model to estimate the risk measures of four types of mortality-dependent financial position: zero-coupon LBs that make a single mortality-dependent payment; coupon-paying LBs that make annual mortality-dependent payments; an annuity book that is hedged by a t-year coupon-paying LB predicated on the same reference population, where t is taken from 1 to 50, and an annuity book that is hedged by a t-year coupon-paying LB predicated on a different reference population. Section 4 explains the risk measures to be estimated—the VaR, ES and spectral risk measures. Section 5 presents two alternative sets of estimates of these risk measures, one ignoring parameter risk, and the other taking account of it. Section 6 concludes.

low, so investors obtained a high coupon rate in return for assuming some degree of exposure to extreme mortality risk.

[5] This instrument was issued by the EIB and managed by BNP Paribas. The face value was £540 million, and involved time t coupon payments that were tied to an initial annuity payment of £50 million indexed to the time t survivor rates of English and Welsh males aged 65 years in 2003. The EIB/BNP bond was closer in spirit to a ‘classic’ longevity bond, because it tied coupon payments to a survivor index and dealt with likely mortality risks rather than extreme ones.

2. A stochastic mortality model

Let $S(t, x)$ be the survivor rate at time t of a cohort aged x in year 0. For any given x, $S(0, x) = 1$ and we expect $S(t, x)$ to diminish as t gets bigger and eventually go to 0 as t gets very large. We also know that if $q(t, x)$ is the realized mortality rate in year $t+1$ (that is, from time t to time $t+1$) of our cohort, then

$$S(t+1, x) = (1 - q(t, x))S(t, x) \tag{1}$$

We assume that $q(t, x)$ is governed by the following two-factor Perks stochastic process:

$$q(t, x) = \frac{e^{A_1(t+1)+A_2(t+1)(t+x)}}{1 + e^{A_1(t+1)+A_2(t+1)(t+x)}} \tag{2}$$

where $A_1(t+1)$ and $A_2(t+1)$ are themselves stochastic processes that are measurable at time $t+1$ (see Perks, 1932; Benjamin and Pollard, 1993). Cairns et al. (2006b) generate empirical results showing that this mortality model provides a good fit to realized male mortality data in England and Wales. Their results also indicate that a two-factor model of UK mortality fits the data better than a one-factor one.

Now let $A(t) = (A_1(t), A_2(t))'$ and assume that $A(t)$ is a random walk with drift:

$$A(t+1) = A(t) + \mu + CZ(t+1) \tag{3}$$

where μ is a constant 2×1 vector of drift parameters, C the constant 2×2 lower triangular matrix reflecting volatilities and correlations, and $Z(t)$ is a 2×1 vector of independent standard normal variables. Cairns et al. (2006b) also show that if we use the UK Government Actuary’s Department (GAD) data for English and Welsh males over 1961–2002, then the least squares estimates of our parameters are:

$$\hat{\mu} = \begin{bmatrix} -0.04340 \\ 0.000367 \end{bmatrix} \tag{4a}$$

$$\hat{V} = \hat{C}\hat{C}' = \begin{bmatrix} 0.01067000, -0.00016170 \\ -0.00016170, 0.00000259 \end{bmatrix} \tag{4b}$$

We can recover $\hat{C}$ from $\hat{V}$ using a Choleski decomposition, and all that remains is to specify a suitable starting value $A(0)$. The results of Cairns et al. suggest that we might take $A(0) \approx (-11.0, 0.107)'$ if we take 2003 as our starting point (i.e., if we set $t = 0$ for the end of 2003).[6]

[6] To elaborate: Fig. 2 in Cairns et al. (2006b) plots estimated $A(0)$ values against time for the years 1961–2002, and the values $A(0) \approx (-11.0, 0.107)'$ for the year 2003 are obtained by simple extrapolation.

Having specified the model, we simulate paths for $A(t)$ over each of $t = 1, 2, \ldots, 50$, using our assumed values of $A(0)$. Each path of $A(t)$ values gives us a path of realized mortality rates $q(t, x)$, and each such path gives us a path for the survivor rates $S(t, x)$.

3. Some mortality-dependent positions

We now consider four types of mortality-dependent position:

- The first and simplest is a long position in a zero-coupon LB (denoted a zero LB below), where a single payment is made in period t and we consider zero LBs for each of $t = 1, 2, \ldots, 50$ in turn. The payment made on this instrument is assumed to be equal to the survivor rate itself, $S(t, x)$. The analysis in this case closely mimics that in Cairns et al. (2006b), except that we are now using different measures of risk.
- The second position is a long position in a coupon-paying LB (denoted coupon LB below), which pays coupons of $S(\tau, x)$ at each time τ from 1 to t, where we again consider each of $t = 1, 2, \ldots, 50$. Thus, the coupon LB makes t payments which end at time t, whereas the zero LB pays just the one payment at time t.
- The third position is an annuity book that is hedged with a coupon LB predicated on the same reference population (that is, hedged with the second position above). The annuity book itself is a commitment to pay $S(t, x)$ in each of $t = 1, 2, \ldots, 50$. As such, the annuity book is equivalent to a short position in a coupon LB with a maturity of 50 years. We assume that this book is combined with a t-maturity coupon LB based on the same reference population, which pays coupons of $S(\tau, x)$ at each time τ from 1 to t, where we again consider each of $t = 0, 1, \ldots, 50$. Of course, the effectiveness of the hedge depends on t, and the hedge should be perfect for $t = 50$.
- The final position is an annuity book hedged by a coupon LB predicated on a different reference population. The difference between the two reference populations suggests that we might expect the hedge to be less effective than before on account of the basis risk it faces, and therefore sometimes produce larger risk measures. This final position addresses the important practical concern that in real-world situations LB hedges would often be predicated on different reference populations, and we would want to know the impact of such differences on the effectiveness of LB hedges.

For each position, we simulate the profit/loss as equal to the simulated future values of the position minus the current ($t = 0$) value of the position.[7] The former are obtained by simulating $S(t, x)$ as explained above and then applying the position payoff function: for example, the payoff for a t-year zero LB is $S(t, x)$, etc. The latter are obtained using the risk-neutral pricing approach of Cairns et al. (2006b).[8] This boils down to replacing the real drift μ with the risk-neutral drift $\mu - C\lambda$, where $\lambda = (\lambda_1, \lambda_2)'$ are the market prices of risk associated with the processes $Z_1(t)$ and $Z_2(t)$. For illustrative purposes, we set λ to the empirically plausible values $(0.175, 0.175)'$.[9] The initial values are then equal to the expected value of the position's future payments discounted at the risk-free rate, with the expectation taken under the risk-neutral probability measure.

4. Measures of financial risk

For each position, we consider three different risk measures. Let q_α be the α quantile of the present value of the loss distribution (where losses are given positive values and profits are given negative ones). The first risk measure is the value-at-risk (VaR), and the VaR at the α confidence level – the αVaR for short – is simply equal to q_α.

The second risk measure is the expected shortfall (ES), and the αES is the average of the worst $1 - \alpha$ losses:

$$\alpha\text{ES} = \frac{1}{1-\alpha}\int_\alpha^1 q_p \, \mathrm{d}p. \tag{5}$$

Our third risk measure is a spectral risk measure (SRM). An SRM is a weighted average of the whole loss distribution:

$$M_\phi = \int_0^1 \phi(p) q_p \, \mathrm{d}p \tag{6}$$

where the weighting function, $\phi(p)$, is given by the user's risk-aversion function. Provided this function satisfies the conditions of non-negativity (i.e., $\phi(p) \geq 0$ for all p belonging to the range [0, 1]), normalization (i.e.,

[7] All values are in discounted terms, and we assume that all discounting is done at a single constant risk-free rate.

[8] Strictly speaking, their approach is a risk-neutral one in the sense that under Q all assets have expected growth rates equal to the risk-free rate. However, we do not assume that markets are complete.

[9] These values are empirically plausible in that they lead to 'sensible' values for the EIB/BNP bond (Cairns et al., 2006b). They are also as reasonable as any others considered in that article. However, the same article also shows that changes in the λ values have relatively little impact on the value of the EIB/BNP bond, i.e., so any 'reasonable' λ values should produce much the same results.

$\int_0^1 \phi(p)\,\mathrm{d}p = 1$) and increasingness (i.e., $\phi(p)$ never falls as p increases), then (6) provides a risk measure that is known to be coherent (see Acerbi, 2004, Proposition 3.4). We now specify a suitable risk-aversion function, and a plausible choice is an exponential risk-aversion function:

$$\phi_k(p) = \frac{k\,\mathrm{e}^{-(1-p)k}}{1-\mathrm{e}^{-k}} \tag{7}$$

where $k \in (0, \infty)$ is the user's degree of absolute risk aversion (ARA). This function satisfies the conditions required of a coherent spectral risk measure, but is also attractive because it is a simple function that depends on a single parameter, the coefficient of absolute risk aversion. To obtain our spectral measure M_ϕ using the exponential weighting function, we choose a value of k and substitute (7) into (6) to get:

$$M_\phi = \int_0^1 \frac{k\,\mathrm{e}^{-(1-p)k}}{1-\mathrm{e}^{-k}} q_p\,\mathrm{d}p \tag{8}$$

The exponential spectral measure is thus a weighted average of quantiles, with the weights given by the exponential risk-aversion function (7).[10]

The weighting functions of the two coherent risk measures are shown in Fig. 1. The ES has a step function that takes a zero value for cumulative probabilities that are less than the confidence level, and a fixed value of 10 for probabilities greater than or equal to the confidence level. The spectral risk measure has a weighting function that is an exponential function of the cumulative probability, and therefore gives loss observations a continuously rising weight as the cumulative probability gets bigger.

We can also say that each of the three risk measures is predicated on a key conditioning parameter. This parameter is the confidence level in the case of the VaR and ES, and the ARA in the case of the spectral measure, and the risk measure will (typically) rise as its conditioning parameter increases. We also know that for any given confidence level, the ES will be above the VaR, but the size of the SRM relative to the VaR and ES will depend on the relative values of the confidence level and the ARA coefficient.

[10] The class of spectral risk measures includes the ES as a special case, obtained by giving tail losses the same weight and other observations a zero weight. It is also very closely related to the class of distortion risk measures introduced by Shaun Wang that have become widely used in the actuarial risk literature (see, e.g., Wang, 1996, 2000). The relationship between these measures is discussed further in Dowd (2005, Chapter 3).

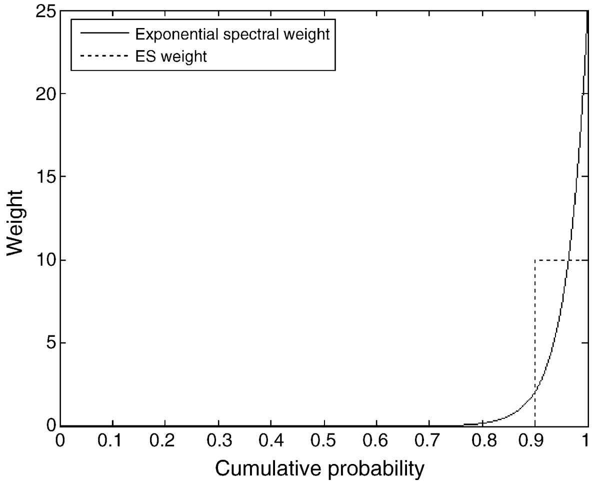

Fig. 1. Weighting functions for coherent risk measures. *Notes*: The ES has the weighting function implied by Eq. (5) with $\alpha = 0.90$, and the exponential spectral risk measure has the weighting function (7) with ARA = 25.

5. Results

For the purposes of producing some illustrative results, we now assume that the discount rate is 4%, that our cohort is aged 65 in year 0, that the VaR and ES are predicated on an arbitrary confidence level of 90%, and that the spectral risk measure is predicated on a coefficient of absolute risk aversion arbitrarily set to 25. For our fourth mortality position, we also assume that the hedge is an LB predicated on a 60-year-old male reference population, instead of the 65-year-old population assumed elsewhere: this ensures that the hedge is predicated on a different reference population to the annuity book.

We should also recall that the risk measures are all denominated in terms of losses, and the amount at risk on a position should be bounded above by the value of the position itself.

5.1. Results without parameter uncertainty

5.1.1. Zero-coupon longevity bonds

As noted earlier, this case is very similar to that of Cairns et al. (2006b), and the only substantive difference is in the risk measures used. Estimated risk measures for a zero LB are given in Table 1 and plotted in Fig. 2. (All positions are considered on a stand-alone basis, without reference to any existing positions.) These give estimates of our three chosen risk measures – the VaR, the ES and the SRM – for zero LBs over maturities ranging from 1 to 50 years. The ES and SRM measures are a

Table 1
Estimated risk measures for zero-coupon longevity bond, with no parameter uncertainty

Coupon year (t)	Initial value	90% VaR	90% ES	SRM (ARA = 25)
1	0.9446	0.0006	0.0008	0.0008
2	0.8910	0.0013	0.0018	0.0020
3	0.8391	0.0023	0.0030	0.0033
4	0.7888	0.0033	0.0045	0.0050
5	0.7400	0.0045	0.0062	0.0068
6	0.6927	0.0060	0.0081	0.0089
7	0.6469	0.0075	0.0101	0.0112
8	0.6025	0.0093	0.0123	0.0136
9	0.5594	0.0111	0.0146	0.0161
10	0.5177	0.0130	0.0170	0.0187
11	0.4774	0.0148	0.0195	0.0214
12	0.4385	0.0165	0.0220	0.0242
13	0.4009	0.0184	0.0244	0.0268
14	0.3647	0.0202	0.0267	0.0294
15	0.3300	0.0218	0.0289	0.0319
16	0.2967	0.0234	0.0310	0.0341
17	0.2650	0.0245	0.0327	0.0359
18	0.2350	0.0258	0.0341	0.0374
19	0.2066	0.0266	0.0351	0.0384
20	0.1799	0.0268	0.0355	0.0389
21	0.1552	0.0266	0.0353	0.0387
22	0.1323	0.0262	0.0346	0.0379
23	0.1115	0.0253	0.0333	0.0364
24	0.0926	0.0240	0.0314	0.0343
25	0.0759	0.0224	0.0290	0.0316
26	0.0612	0.0204	0.0263	0.0284
27	0.0485	0.0184	0.0232	0.0250
28	0.0378	0.0162	0.0201	0.0215
29	0.0288	0.0138	0.0169	0.0180
30	0.0215	0.0114	0.0138	0.0146
31	0.0157	0.0091	0.0109	0.0115
32	0.0112	0.0072	0.0084	0.0087
33	0.0078	0.0054	0.0062	0.0064
34	0.0053	0.0040	0.0045	0.0046
35	0.0035	0.0028	0.0031	0.0032
36	0.0023	0.0019	0.0021	0.0021
37	0.0014	0.0013	0.0013	0.0014
38	0.0009	0.0008	0.0008	0.0009
39	0.0005	0.0005	0.0005	0.0005
40	0.0003	0.0003	0.0003	0.0003
41	0.0002	0.0002	0.0002	0.0002
42	0.0001	0.0001	0.0001	0.0001
43	0.0001	0.0001	0.0001	0.0001
44	0	0	0	0
45	0	0	0	0
46	0	0	0	0
47	0	0	0	0
48	0	0	0	0
49	0	0	0	0
50	0	0	0	0

Notes: Estimated using 5000 Monte Carlo simulation trials. The model is calibrated using male mortality data for England and Wales over period 1961–2002 using the parameter values in (4a) and (4b), with initial values of $A_1(0) = -11.0$ and $A_2(0) = 0.107$ for $t = 0$ taken as 2003. The age at time 0 is 65 years, the discount rate is 4%, and the values of λ_1 and λ_2 are both set to 0.175. The third and fourth columns give the estimated VaR and ES at the 90% confidence level, and the fifth gives estimates of the spectral risk measure (8) with an absolute risk aversion coefficient of 25.

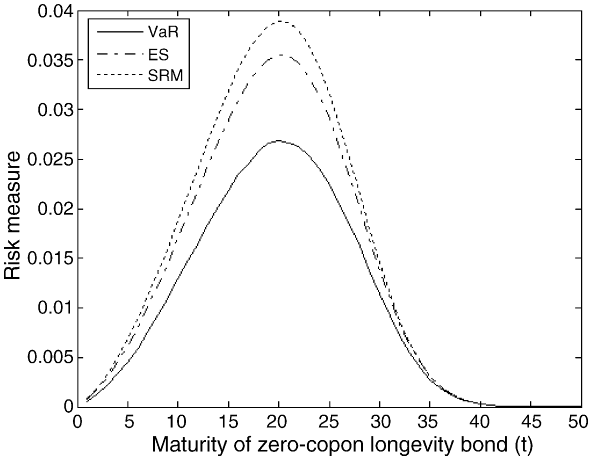

Fig. 2. Estimated absolute risk measures for a zero-coupon longevity bond, no parameter uncertainty. *Notes*: As per Table 1.

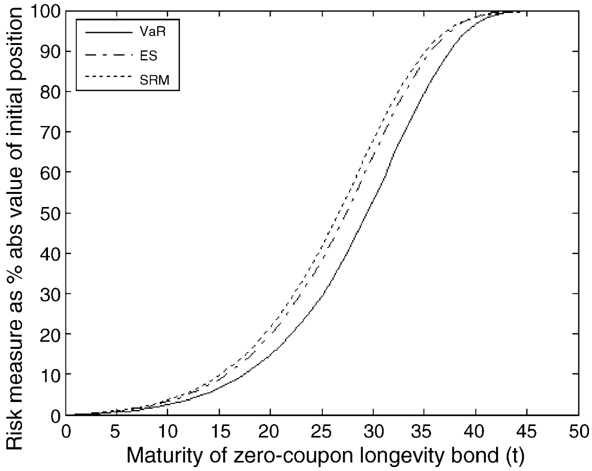

Fig. 3. Estimated relative risk measures for a zero-coupon longevity bond, no parameter uncertainty. *Notes*: As per Table 1.

little higher than the VaRs, but estimates of all three risk measures paint a similar picture: the zeros have relatively low risk over short horizons, have relatively high risk over medium-term horizons, and have relatively low risk over long horizons. In this case, the peak risk occurs over a horizon of about 20 years, and it is interesting to note that there is a considerable amount of long-term risk: for example, the risk measures of a 30-year zero are comparable to those of a 10-year zero.

The explanation for this behavior is that the risk measures reflect two offsetting random effects. The first of these is a diffusion effect, which arises because longer term survivor rates are more uncertain than shorter term survivor rates; this stems from the fact that longer term survivor rates are subject to more mortality shocks. This diffusion effect grows with the horizon. The second effect is an amortization effect arising from the facts that payments are linked to survivor rates, and survivor rates decrease over time towards zero. This amortization effect serves to reduce risk measures, and becomes stronger as the horizon gets longer. Initially, the diffusion effect is dominant and the estimated risk measures grow with t. But as t rises, the amortization effect comes into play and serves to curtail the increase in risk. Eventually, the amortization effect becomes dominant, and the risk measures fall back again. In the limit, the survivor rate goes to zero and the estimated risk measures go to zero as well.

However, Fig. 2 only gives the risk measures in absolute (i.e., monetary) terms, and these can give a misleading impression of relative risks because the issue values of these bonds are very different. To take account of these differences, Fig. 3 shows the relative risk measures—the risk measures expressed as percentages of investments in these bonds. In effect, these relative risk measures strip out the amortization effect, and reflect only the impact of diffusion. It is therefore not surprising that Fig. 3 shows that relative risks rise with t. For example, an investment in 15-year zero LBs has a VaR that is about 5% of the amount invested, whereas an investment in 35-year zero LBs has a VaR that is over 70% of the value of the amount invested. (For their part, the ES and SRM estimates are a little higher than the VaR ones.) Thus, an investment in longer term zero LBs is more risky (and, potentially, a lot more risky) than an investment in short-term zero LBs.

5.1.2. *Coupon-paying longevity bonds*

Comparable results for coupon LBs are given in Figs. 4 and 5 and Table 2. The curves for these risk mea-

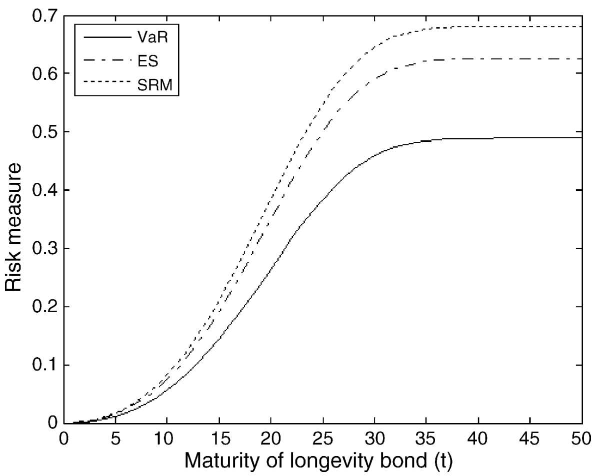

Fig. 4. Estimated absolute risk measures for a coupon-paying longevity bond, no parameter uncertainty. *Notes*: As per Table 2.

Table 2
Estimated absolute risk measures for a coupon-paying longevity bond, no parameter uncertainty

Coupon year (*t*)	Initial value	90% VaR	90% ES	SRM (ARA = 25)
1	0.9446	0.0006	0.0008	0.0008
2	1.8357	0.0019	0.0025	0.0027
3	2.6747	0.0040	0.0054	0.0060
4	3.4635	0.0073	0.0097	0.0108
5	4.2035	0.0117	0.0156	0.0174
6	4.8962	0.0173	0.0234	0.0260
7	5.5431	0.0244	0.0331	0.0368
8	6.1455	0.0334	0.0449	0.0499
9	6.7049	0.0442	0.0589	0.0654
10	7.2227	0.0568	0.0752	0.0834
11	7.7001	0.0709	0.0938	0.1038
12	8.1385	0.0873	0.1148	0.1268
13	8.5394	0.1044	0.1381	0.1522
14	8.9041	0.1231	0.1636	0.1801
15	9.2341	0.1436	0.1911	0.2102
16	9.5309	0.1659	0.2204	0.2423
17	9.7959	0.1892	0.2514	0.2761
18	10.0308	0.2141	0.2835	0.3112
19	10.2374	0.2372	0.3163	0.3471
20	10.4173	0.2626	0.3494	0.3834
21	10.5725	0.2899	0.3821	0.4193
22	10.7048	0.3153	0.4144	0.4543
23	10.8162	0.3399	0.4453	0.4878
24	10.9089	0.3642	0.4742	0.5192
25	10.9848	0.3841	0.5008	0.5480
26	11.0460	0.4032	0.5247	0.5739
27	11.0946	0.4216	0.5458	0.5966
28	11.1323	0.4369	0.5638	0.6160
29	11.1611	0.4481	0.5789	0.6321
30	11.1826	0.4586	0.5912	0.6452
31	11.1984	0.4676	0.6009	0.6554
32	11.2096	0.4732	0.6084	0.6632
33	11.2174	0.4774	0.6139	0.669
34	11.2227	0.4818	0.6179	0.6731
35	11.2262	0.4844	0.6206	0.6759
36	11.2285	0.4862	0.6225	0.6778
37	11.2300	0.4874	0.6237	0.6791
38	11.2308	0.4882	0.6245	0.6799
39	11.2314	0.4887	0.6249	0.6803
40	11.2317	0.4890	0.6252	0.6806
41	11.2319	0.4891	0.6254	0.6808
42	11.2320	0.4892	0.6255	0.6809
43	11.2320	0.4892	0.6256	0.6810
44	11.2321	0.4893	0.6256	0.6810
45	11.2321	0.4893	0.6256	0.6810
46	11.2321	0.4893	0.6256	0.6810
47	11.2321	0.4893	0.6256	0.6810
48	11.2321	0.4893	0.6256	0.6810
49	11.2321	0.4893	0.6256	0.6810
50	11.2321	0.4893	0.6256	0.6810

Notes: Estimated using 5000 Monte Carlo simulation trials. The longevity bond pays a coupon each year (i) equal to the survivor rate of the original reference population still alive at i, for all i up to t. Other details are as given in the notes to Table 1.

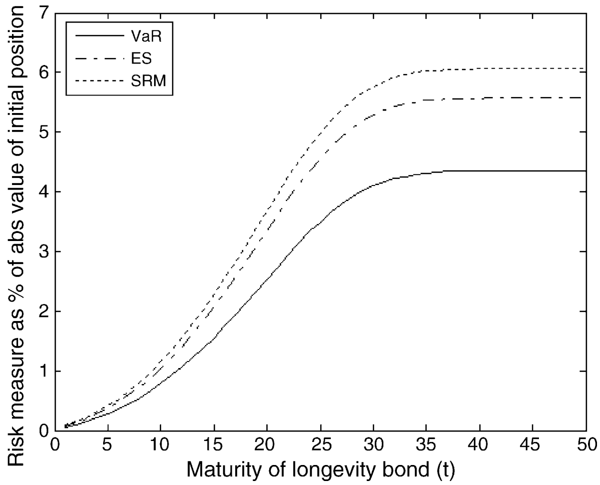

Fig. 5. Estimated relative risk measures for a coupon-paying longevity bond, no parameter uncertainty. *Notes*: As per Table 2.

sures have a similar shape and again reflect the impact of diffusion and amortization effects. In all cases, the risk curve has a starting value of zero and then rises in a logistic fashion to plateau when t reaches a value of about 40. The plateaus in Fig. 4 are about 0.5 for the VaR and between 0.6 and 0.7 for the ES and SRM, and the plateaus in Fig. 5 are just over 4% for the VaR and around 6% for the other two risk measures. The fact that the estimated risk measures peak at long horizons indicates that there is a lot of risk exposure arising specifically from the longer end of the mortality term structure: for example, a 30-year LB is considerably more risky than, say, a 15-year one.

A comparison of Figs. 3 and 5 also shows that for any given t and position value, a coupon LB has considerably lower risk than a zero LB. The reason for this is that the coupon LB involves a more diversified mortality exposure than a zero LB. In quantitative terms, this comparison also indicates that the coupon LB can have a *much* lower estimated risk than the zero LB, and that this difference is especially pronounced for higher maturity instruments.

5.1.3. Annuity book with an LB hedge based on the same reference population

Fig. 6 and Table 3 give results for our third position, the annuity book hedged by a LB predicated on the same reference population. Given the nature of the hedge instrument, the hedge eliminates all exposure up to maturity $t+1$. This leads us to expect that, as t gets bigger, more and more risk is being hedged so estimated risk measures should get smaller. Furthermore, when t

Table 3
Estimated risk measures for an annuity book hedged with a coupon-paying longevity bond based on the same population cohort, no parameter uncertainty

Maturity of hedge instrument	Initial value of hedged position	90% VaR	90% ES	SRM (ARA = 25)
1	−10.2875	0.3794	0.5644	0.6449
2	−9.3964	0.3800	0.5641	0.6445
3	−8.5573	0.3795	0.5633	0.6437
4	−7.7686	0.3776	0.5621	0.6422
5	−7.0286	0.3793	0.5601	0.6400
6	−6.3359	0.3779	0.5573	0.6368
7	−5.6890	0.3770	0.5535	0.6325
8	−5.0865	0.3736	0.5485	0.6268
9	−4.5271	0.3676	0.5421	0.6197
10	−4.0094	0.3626	0.5343	0.6109
11	−3.5320	0.3552	0.5248	0.6002
12	−3.0935	0.3485	0.5135	0.5876
13	−2.6927	0.3376	0.5002	0.5730
14	−2.3280	0.3272	0.4850	0.5561
15	−1.9980	0.3157	0.4676	0.5370
16	−1.7012	0.3032	0.4482	0.5155
17	−1.4362	0.2874	0.4269	0.4918
18	−1.2012	0.2692	0.4036	0.4659
19	−0.9947	0.2524	0.3786	0.4381
20	−0.8148	0.2353	0.3522	0.4086
21	−0.6596	0.2162	0.3246	0.3776
22	−0.5273	0.1954	0.2963	0.3457
23	−0.4158	0.1759	0.2676	0.3130
24	−0.3232	0.1558	0.2387	0.2803
25	−0.2473	0.1350	0.2102	0.2479
26	−0.1861	0.1158	0.1826	0.2164
27	−0.1375	0.0975	0.1563	0.1862
28	−0.0998	0.0796	0.1317	0.1579
29	−0.0710	0.0644	0.1091	0.1318
30	−0.0494	0.0509	0.0887	0.1083
31	−0.0337	0.0390	0.0708	0.0874
32	−0.0225	0.0300	0.0554	0.0693
33	−0.0147	0.0219	0.0424	0.0540
34	−0.0094	0.0157	0.0319	0.0413
35	−0.0059	0.0108	0.0234	0.0310
36	−0.0036	0.0072	0.0169	0.0228
37	−0.0021	0.0046	0.0119	0.0164
38	−0.0013	0.0029	0.0082	0.0116
39	−0.0007	0.0017	0.0055	0.0081
40	−0.0004	0.0010	0.0036	0.0055
41	−0.0002	0.0005	0.0024	0.0037
42	−0.0001	0.0003	0.0015	0.0024
43	−0.0001	0.0001	0.0009	0.0015
44	0	0.0001	0.0005	0.0010
45	0	0	0.0003	0.0006
46	0	0	0.0002	0.0003
47	0	0	0.0001	0.0002
48	0	0	0	0.0001
49	0	0	0	0
50	0	0	0	0

Notes: Estimated using 5000 Monte Carlo simulation trials. The results in the table refer to a position in an annuity book hedged by a coupon longevity bond of maturity *t*, where both the annuity book and the bond make payments equal to the survivor rate of the original reference population. Other details are as given in Table 1.

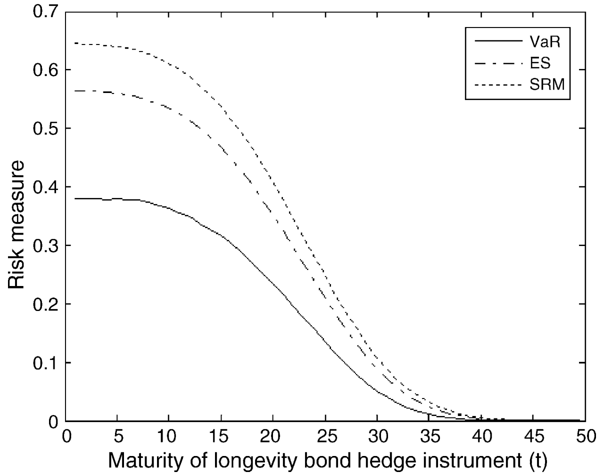

Fig. 6. Estimated absolute risk measures for an annuity book hedged with a coupon-paying longevity bond based on the same population cohort, no parameter uncertainty. *Notes*: As per Table 3.

gets very large, there should be little risk exposure left, and the estimated risk measures should approach 0.

When we look at our results, we see that these expectations are broadly met for all three risk measures. In addition, the fact that the curves continue to fall after $t = 25$ confirms our earlier conclusion that there is considerable mortality exposure arising from the longer end of the mortality term structure.

These findings also tells us that even if the hedge is predicated on the same reference population as the insurer's annuity portfolio, a 25-year longevity bond still leaves a considerable net exposure. Thus, it is clear not only that longer maturity LBs would provide better hedges for annuity books than shorter maturity LBs, but

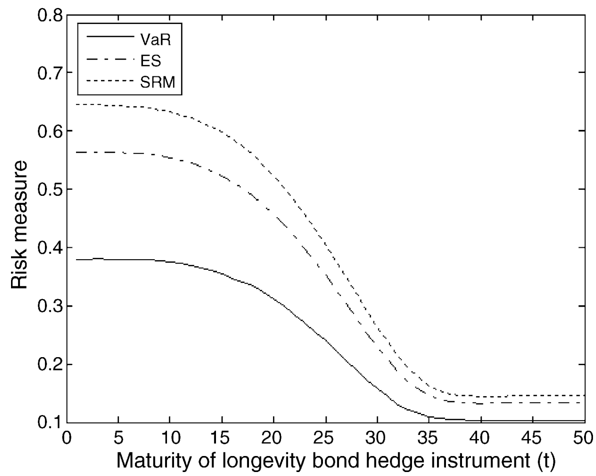

Fig. 7. Estimated absolute risk measures for an annuity book hedged with a coupon-paying longevity bond based on a different population cohort, no parameter uncertainty. *Notes*: As per Table 4.

Table 4
Estimated risk measures for an annuity book hedged with a coupon-paying longevity bond based on a different population cohort, no parameter uncertainty

Maturity of hedge instrument	Initial value of hedged position	90% VaR	90% ES	SRM (ARA = 25)
1	−10.2808	0.3796	0.5645	0.6450
2	−9.3766	0.3798	0.5644	0.6449
3	−8.518	0.3805	0.5642	0.6447
4	−7.7034	0.3802	0.5639	0.6442
5	−6.9315	0.3785	0.5633	0.6435
6	−6.2010	0.3803	0.5624	0.6424
7	−5.5105	0.3790	0.5611	0.6408
8	−4.8589	0.3787	0.5593	0.6387
9	−4.2450	0.3781	0.5569	0.6359
10	−3.6677	0.3754	0.5538	0.6323
11	−3.1259	0.3740	0.5498	0.6277
12	−2.6186	0.3697	0.5449	0.6221
13	−2.1448	0.3647	0.5389	0.6153
14	−1.7035	0.3602	0.5317	0.6071
15	−1.2937	0.3550	0.5232	0.5975
16	−0.9145	0.3474	0.5132	0.5862
17	−0.5650	0.3407	0.5017	0.5733
18	−0.2441	0.3341	0.4886	0.5586
19	0.0490	0.3241	0.4738	0.5420
20	0.3154	0.3110	0.4574	0.5235
21	0.5560	0.2987	0.4393	0.5032
22	0.7721	0.2853	0.4196	0.4810
23	0.9646	0.2712	0.3984	0.4570
24	1.1348	0.2555	0.3759	0.4315
25	1.2840	0.2393	0.3522	0.4046
26	1.4135	0.2231	0.3277	0.3768
27	1.5247	0.2059	0.3028	0.3483
28	1.6191	0.1881	0.2776	0.3196
29	1.6981	0.1721	0.2529	0.2911
30	1.7634	0.1582	0.2291	0.2636
31	1.8164	0.1443	0.2069	0.2376
32	1.8589	0.1306	0.1868	0.2138
33	1.8922	0.1211	0.1695	0.1930
34	1.9178	0.1146	0.1556	0.1758
35	1.9372	0.1095	0.1457	0.1629
36	1.9516	0.1076	0.1393	0.1540
37	1.9619	0.1051	0.1355	0.1487
38	1.9692	0.1048	0.1335	0.1461
39	1.9743	0.1041	0.1328	0.1450
40	1.9777	0.1031	0.1327	0.1448
41	1.9799	0.1031	0.1328	0.1448
42	1.9813	0.1032	0.1330	0.1449
43	1.9822	0.1034	0.1332	0.1451
44	1.9828	0.1033	0.1333	0.1452
45	1.9831	0.1035	0.1334	0.1452
46	1.9833	0.1037	0.1334	0.1453
47	1.9834	0.1037	0.1334	0.1453
48	1.9835	0.1036	0.1335	0.1453
49	1.9836	0.1036	0.1335	0.1453
50	1.9836	0.1036	0.1335	0.1453

Notes: Estimated using 5000 Monte Carlo simulation trials. The results in the table refer to a position in an annuity book hedged by a coupon longevity bond of maturity *t*, where the annuity book makes a payment equal to the survivor rate of the original reference population, and the bond makes a payment equal to the survivor rate of a 60-year original reference population. Other details are as given in Table 1.

Table 5
Estimated risk measures for a zero-coupon longevity bond, with parameter uncertainty

Coupon year (t)	Initial value	90% VaR	90% ES	SRM (ARA = 25)
1	0.9446	0.0006	0.0008	0.0009
2	0.8910	0.0014	0.0019	0.0021
3	0.8391	0.0024	0.0033	0.0037
4	0.7888	0.0036	0.0049	0.0055
5	0.7401	0.0051	0.0068	0.0076
6	0.6929	0.0066	0.0090	0.0100
7	0.6472	0.0084	0.0113	0.0127
8	0.6029	0.0102	0.0138	0.0155
9	0.5600	0.0123	0.0165	0.0185
10	0.5186	0.0143	0.0193	0.0216
11	0.4785	0.0163	0.0222	0.0248
12	0.4399	0.0183	0.0251	0.0281
13	0.4027	0.0204	0.0280	0.0314
14	0.3669	0.0226	0.0308	0.0345
15	0.3326	0.0245	0.0335	0.0375
16	0.2999	0.0264	0.0360	0.0402
17	0.2687	0.0280	0.0382	0.0425
18	0.2391	0.0291	0.0400	0.0445
19	0.2113	0.0301	0.0414	0.0460
20	0.1852	0.0309	0.0422	0.0468
21	0.1608	0.0313	0.0424	0.0470
22	0.1384	0.0314	0.0420	0.0463
23	0.1179	0.0307	0.0408	0.0448
24	0.0993	0.0298	0.0389	0.0426
25	0.0827	0.0281	0.0364	0.0396
26	0.0680	0.0261	0.0334	0.0362
27	0.0552	0.0239	0.0301	0.0323
28	0.0442	0.0215	0.0265	0.0283
29	0.0348	0.0188	0.0228	0.0242
30	0.0271	0.0161	0.0192	0.0202
31	0.0207	0.0135	0.0157	0.0164
32	0.0156	0.0111	0.0126	0.0131
33	0.0116	0.0089	0.0099	0.0101
34	0.0085	0.0069	0.0076	0.0077
35	0.0061	0.0053	0.0056	0.0057
36	0.0044	0.0039	0.0041	0.0042
37	0.0031	0.0029	0.0030	0.0030
38	0.0022	0.0021	0.0021	0.0021
39	0.0015	0.0014	0.0015	0.0015
40	0.0010	0.0010	0.0010	0.0010
41	0.0007	0.0007	0.0007	0.0007
42	0.0005	0.0005	0.0005	0.0005
43	0.0003	0.0003	0.0003	0.0003
44	0.0002	0.0002	0.0002	0.0002
45	0.0002	0.0002	0.0002	0.0002
46	0.0001	0.0001	0.0001	0.0001
47	0.0001	0.0001	0.0001	0.0001
48	0.0001	0.0001	0.0001	0.0001
49	0	0	0	0
50	0	0	0	0

Notes: As per Table 1, except for V being simulated from the inverse of a normal Wishart($n-1, n^{-1}\hat{V}^{-1}$) distribution and μ being simulated from a MVN($\hat{\mu}, n^{-1}V$) along the lines explained in Appendix A.

Table 6
Estimated risk measures for a coupon-paying longevity bond, with parameter uncertainty

Coupon year (t)	Initial value	90% VaR	90% ES	SRM (ARA = 25)
1	0.9446	0.0006	0.0008	0.0009
2	1.8357	0.0019	0.0027	0.0030
3	2.6748	0.0043	0.0058	0.0066
4	3.4636	0.0078	0.0106	0.0119
5	4.2037	0.0127	0.0172	0.0193
6	4.8966	0.0190	0.0259	0.0290
7	5.5437	0.0270	0.0368	0.0412
8	6.1466	0.0369	0.0503	0.0562
9	6.7066	0.0490	0.0662	0.0741
10	7.2252	0.0626	0.0848	0.0949
11	7.7037	0.0789	0.1061	0.1188
12	8.1436	0.0970	0.1303	0.1458
13	8.5462	0.1164	0.1572	0.1758
14	8.9132	0.1375	0.1866	0.2088
15	9.2458	0.1606	0.2187	0.2446
16	9.5457	0.1850	0.2532	0.2830
17	9.8144	0.2125	0.2896	0.3235
18	10.0536	0.2420	0.3277	0.3658
19	10.2648	0.2719	0.3671	0.4094
20	10.4500	0.3014	0.4071	0.4537
21	10.6108	0.3305	0.4472	0.4980
22	10.7492	0.3612	0.4867	0.5415
23	10.8671	0.3865	0.5251	0.5836
24	10.9664	0.4155	0.5616	0.6236
25	11.0491	0.4408	0.5956	0.6607
26	11.1171	0.4651	0.6269	0.6946
27	11.1723	0.4870	0.6551	0.7248
28	11.2164	0.5077	0.6798	0.7512
29	11.2513	0.5251	0.7010	0.7737
30	11.2783	0.5403	0.7188	0.7925
31	11.2990	0.5524	0.7335	0.8078
32	11.3147	0.5622	0.7452	0.8200
33	11.3262	0.5713	0.7544	0.8294
34	11.3347	0.5775	0.7614	0.8367
35	11.3409	0.5823	0.7667	0.8421
36	11.3452	0.5864	0.7706	0.8460
37	11.3483	0.5893	0.7734	0.8489
38	11.3505	0.5913	0.7754	0.8509
39	11.3520	0.5927	0.7768	0.8523
40	11.3530	0.5937	0.7778	0.8533
41	11.3537	0.5944	0.7785	0.8540
42	11.3542	0.5949	0.7790	0.8545
43	11.3545	0.5952	0.7793	0.8548
44	11.3547	0.5954	0.7795	0.8550
45	11.3549	0.5956	0.7797	0.8552
46	11.3550	0.5957	0.7798	0.8553
47	11.3551	0.5958	0.7799	0.8554
48	11.3551	0.5958	0.7799	0.8554
49	11.3552	0.5959	0.7800	0.8555
50	11.3552	0.5959	0.7800	0.8555

Notes: As per Table 2, except for V being simulated from the inverse of a normal Wishart$(n-1, n^{-1}\hat{V}^{-1})$ distribution and μ being simulated from a MVN$(\hat{\mu}, n^{-1}V)$ along the lines explained in Appendix A.

Table 7
Estimated risk measures for an annuity book hedged with a coupon-paying longevity bond based on same population cohort, with parameter uncertainty

Maturity of hedge instrument	Initial value of hedged position	90% VaR	90% ES	SRM (ARA = 25)
1	−10.4106	0.5254	0.7822	0.9039
2	−9.5195	0.5253	0.7818	0.9034
3	−8.6804	0.5250	0.7810	0.9024
4	−7.8916	0.5252	0.7794	0.9007
5	−7.1515	0.5230	0.7771	0.8981
6	−6.4586	0.5215	0.7737	0.8943
7	−5.8115	0.5180	0.7691	0.8892
8	−5.2086	0.5139	0.7631	0.8825
9	−4.6486	0.5083	0.7555	0.8740
10	−4.1300	0.5011	0.7462	0.8635
11	−3.6515	0.4935	0.7348	0.8508
12	−3.2116	0.4866	0.7213	0.8356
13	−2.8090	0.4744	0.7055	0.8179
14	−2.4420	0.4624	0.6873	0.7976
15	−2.1094	0.4471	0.6666	0.7744
16	−1.8095	0.4285	0.6434	0.7484
17	−1.5408	0.4087	0.6176	0.7197
18	−1.3016	0.3860	0.5895	0.6883
19	−1.0904	0.3621	0.5591	0.6544
20	−0.9052	0.3385	0.5267	0.6183
21	−0.7444	0.3149	0.4925	0.5802
22	−0.6060	0.2883	0.4569	0.5406
23	−0.4881	0.2622	0.4203	0.4998
24	−0.3888	0.2356	0.3833	0.4583
25	−0.3061	0.2090	0.3464	0.4167
26	−0.2381	0.1835	0.3100	0.3756
27	−0.1829	0.1598	0.2747	0.3354
28	−0.1388	0.1349	0.2408	0.2968
29	−0.1039	0.1132	0.2089	0.2601
30	−0.0769	0.0936	0.1792	0.2257
31	−0.0562	0.0760	0.1520	0.1940
32	−0.0406	0.0604	0.1274	0.1651
33	−0.0290	0.0475	0.1057	0.1392
34	−0.0205	0.0371	0.0867	0.1162
35	−0.0143	0.0283	0.0703	0.0962
36	−0.0100	0.0213	0.0564	0.0789
37	−0.0069	0.0156	0.0448	0.0641
38	−0.0047	0.0109	0.0352	0.0517
39	−0.0032	0.0075	0.0274	0.0413
40	−0.0022	0.0052	0.0211	0.0327
41	−0.0015	0.0034	0.0160	0.0256
42	−0.0010	0.0022	0.0121	0.0198
43	−0.0007	0.0013	0.0089	0.0151
44	−0.0005	0.0008	0.0065	0.0113
45	−0.0003	0.0004	0.0046	0.0083
46	−0.0002	0.0002	0.0032	0.0058
47	−0.0001	0.0001	0.0021	0.0039
48	−0.0001	0	0.0012	0.0023
49	0	0	0.0005	0.0010
50	0	0	0	0

Notes: As per Table 3, except for V being simulated from the inverse of a normal Wishart$(n-1, n^{-1}\hat{V}^{-1})$ distribution and μ being simulated from a MVN$(\hat{\mu}, n^{-1}V)$ along the lines explained in Appendix A.

Table 8
Estimated risk measures for annuity book hedged with a coupon longevity bond based on different cohort, with parameter uncertainty

Maturity of hedge instrument	Initial value of hedged position	90% VaR	90% ES	SRM (ARA = 25)
1	−10.4039	0.5253	0.7823	0.9040
2	−9.4997	0.5256	0.7822	0.9038
3	−8.6411	0.5253	0.7820	0.9035
4	−7.8266	0.5250	0.7815	0.9030
5	−7.0548	0.5254	0.7807	0.9020
6	−6.3243	0.5248	0.7795	0.9007
7	−5.6339	0.5237	0.7778	0.8987
8	−4.9824	0.5228	0.7755	0.8961
9	−4.3685	0.5192	0.7724	0.8925
10	−3.7911	0.5175	0.7684	0.8880
11	−3.2492	0.5139	0.7634	0.8822
12	−2.7416	0.5091	0.7573	0.8752
13	−2.2673	0.5031	0.7498	0.8666
14	−1.8253	0.4985	0.7409	0.8564
15	−1.4146	0.4907	0.7303	0.8443
16	−1.0342	0.4843	0.7180	0.8303
17	−0.6831	0.4750	0.7038	0.8142
18	−0.3604	0.4635	0.6876	0.7958
19	−0.0649	0.4502	0.6692	0.7751
20	0.2042	0.4355	0.6487	0.7521
21	0.4481	0.4191	0.6261	0.7267
22	0.6678	0.4017	0.6013	0.6989
23	0.8646	0.3838	0.5745	0.6689
24	1.0396	0.3608	0.5457	0.6366
25	1.1941	0.3398	0.5152	0.6025
26	1.3293	0.3174	0.4833	0.5667
27	1.4466	0.2920	0.4504	0.5296
28	1.5475	0.2683	0.4166	0.4917
29	1.6332	0.2454	0.3826	0.4532
30	1.7054	0.2216	0.3489	0.4149
31	1.7653	0.1999	0.3159	0.3772
32	1.8145	0.1793	0.2844	0.3407
33	1.8545	0.1607	0.2550	0.3062
34	1.8864	0.1458	0.2285	0.2745
35	1.9116	0.1345	0.2056	0.2464
36	1.9312	0.1271	0.1876	0.2227
37	1.9462	0.1223	0.1744	0.2040
38	1.9576	0.1194	0.1651	0.1903
39	1.9662	0.1179	0.1592	0.1812
40	1.9725	0.1169	0.1558	0.1754
41	1.9771	0.1172	0.1541	0.1719
42	1.9804	0.1180	0.1533	0.1696
43	1.9829	0.1184	0.1529	0.1681
44	1.9846	0.1188	0.1526	0.1671
45	1.9858	0.1190	0.1524	0.1665
46	1.9867	0.1190	0.1523	0.1663
47	1.9873	0.1190	0.1523	0.1662
48	1.9877	0.1193	0.1524	0.1661
49	1.9880	0.1192	0.1524	0.1662
50	1.9882	0.1193	0.1525	0.1663

Notes: As per Table 4, except for V being simulated from the inverse of a normal Wishart($n-1, n^{-1}\hat{V}^{-1}$) distribution and μ being simulated from a MVN($\hat{\mu}, n^{-1}V$) along the lines explained in Appendix A.

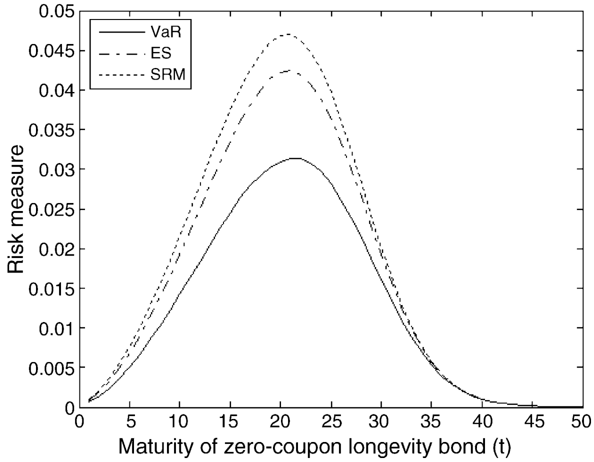

Fig. 8. Estimated absolute risk measures for a zero-coupon longevity bond, with parameter uncertainty. *Notes*: As per Table 1, except for V being simulated from the inverse of a normal Wishart($n-1, n^{-1}\hat{V}^{-1}$) distribution and μ being simulated from a MVN($\hat{\mu}, n^{-1}V$) along the lines explained in Appendix A.

also that LBs with ultra-long maturities would provide even better hedges. In other words, an LB with a 25 years maturity (like the EIB/BNP bond) can provide very significant hedging benefits to an insurer, but a bond with a longer maturity would provide even more.

5.1.4. Annuity book with an LB hedge based on a different reference population

Table 4 and Fig. 7 give the corresponding results for our hedged annuity book where the hedge is based on a different reference population. We find that the difference between the two reference populations does not have much effect for low t-values, but certainly does have

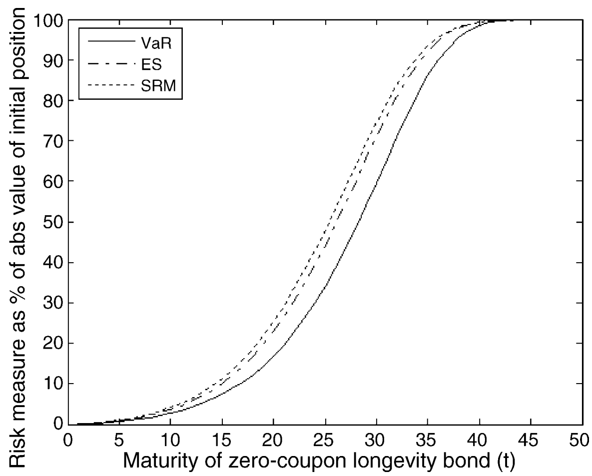

Fig. 9. Estimated relative risk measures for zero-coupon longevity bond, with parameter uncertainty. *Notes*: As per Table 5.

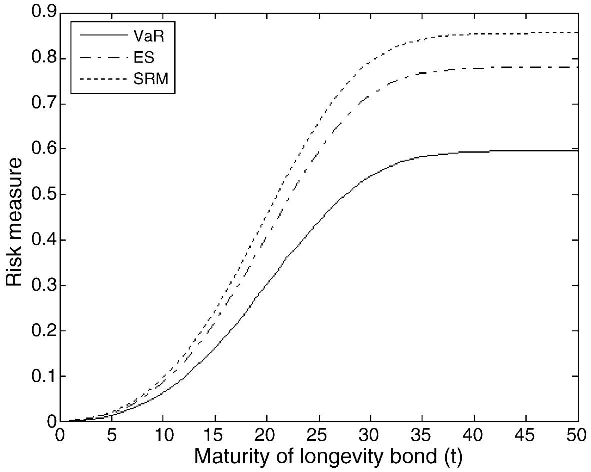

Fig. 10. Estimated absolute risk measures for a coupon-paying longevity bond, with parameter uncertainty. *Notes*: As per Table 6.

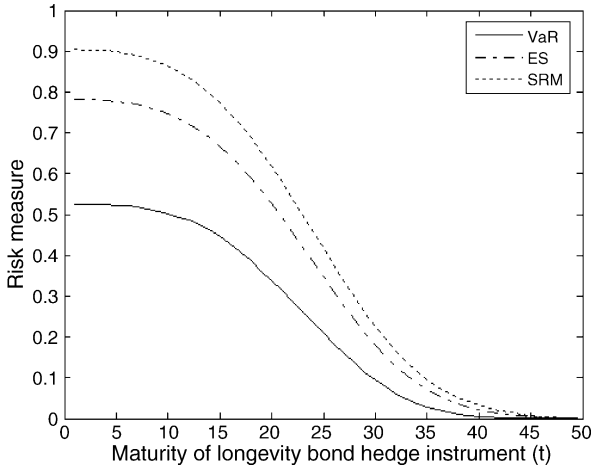

Fig. 12. Estimated absolute risk measures for an annuity book hedged with a coupon-paying longevity bond based on same population cohort, with parameter uncertainty. *Notes*: As per Table 6.

an effect at medium and long-term horizons. This tells us that in this case – unlike the last – even a very long-term hedge instrument can still leave a significant degree of basis risk.

5.2. *Results with parameter uncertainty*

Now suppose that we treat the parameters μ and V as uncertain. We can model this uncertainty using Bayesian methods, and one plausible method involves the use of a non-informative prior such as the Jeffreys prior, which postulates that the distribution of the parameters is proportional to the determinant of V. The Jeffreys prior also implies that the posterior distribution for V follows a normal-inverse Wishart distribution based on an estimate of V, $\hat{V}$, and that μ is multivariate normal with mean $\hat{\mu}$ and variance $n^{-1}V$, where n is the sample size equal here to 41. This allows us to simulate both μ and V on the basis of initial estimates of $\hat{\mu}$ and $\hat{V}$. Details of the simulation algorithm are available in Appendix A.

We are interested here in how taking account of parameter uncertainty affects our earlier results. Our earlier analyses were therefore repeated using simulated parameters instead of our fixed parameters, and the results are presented in Tables 5–8 and Figs. 8–13, which are the analogues of Tables 1–4 and Figs. 2–7. These results indicate that estimated risk measures that take account of parameter uncertainty can be somewhat higher than those that ignore it. However, the increase

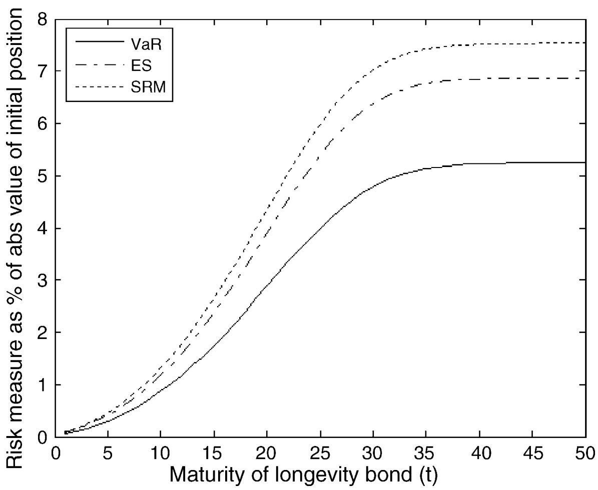

Fig. 11. Estimated relative risk measures for a coupon-paying longevity bond, with parameter uncertainty. *Notes*: As per Table 6.

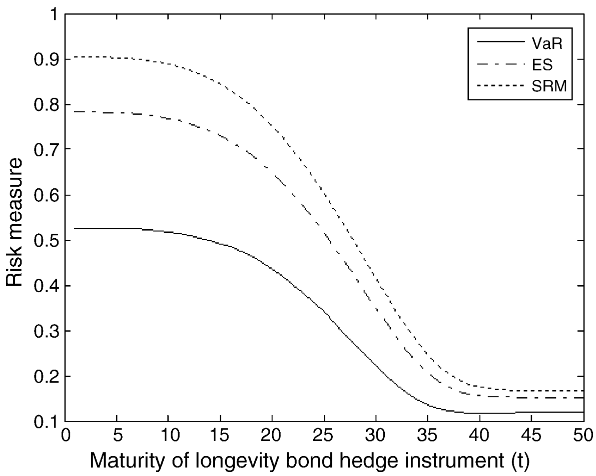

Fig. 13. Estimated absolute risk measures for annuity book hedged with a coupon longevity bond based on different cohort, with parameter uncertainty. *Notes*: As per Table 8.

in estimated risk measures depends on both the type of position and the horizon period:

- For the zero LB, estimated risk measures can be up to about 15% higher.
- For the coupon-paying LB, the plateaus to which the risk measures tend are about 25% higher with parameter uncertainty.
- For the hedged annuity books, we find that for low horizons, the estimated risk measures are about 25% higher with parameter uncertainty, but as the horizon increases the impact of parameter uncertainty on estimated risk measures tends to fall.

6. Conclusions

This paper has applied the recently developed two-factor mortality model of Cairns et al. to estimate financial risk measures for an illustrative set of mortality-dependent positions. The model is calibrated on UK data, and we estimate VaRs, ESs and spectral risk measures for positions in zero-coupon longevity bonds, coupon-paying longevity bonds, and an annuity book hedged by each of two alternative longevity bonds. In each case, we consider maturities of up to 50 years.

Our results suggest that all three risk measures give broadly similar indications of the risks involved. They also suggest that mortality-dependent positions can sometimes be very risky: in absolute (i.e., \$) terms, most of the risk relates to the medium term of around 10–30 years; however, in relative terms (i.e., measuring risk relative to the value of the position) there is often a considerable amount of risk emanating from the long end of the maturity spectrum (i.e., related to the risk of very high longevity). We also present alternative sets of results that do and do not take account of parameter uncertainty, and find that the latter can be somewhat larger than the former, which indicates that taking account of parameter uncertainty often leads to increased estimates of financial risk measures.

Finally, we address the issue of the usefulness of our analysis and of the results they produce. We would suggest that these have the potential to be useful to the insurance industry in two main ways. First, the analysis itself is useful in that it shows how the two-factor mortality model can be used to estimate quantile-based risk measures for a number of mortality-dependent positions. Of course, the positions selected were only illustrative, but it is obvious that the basic approach can be applied to *any* mortality-dependent positions. In other words, the paper provides a blueprint that enables risk managers to estimate risk measures for any of the mortality-dependent risks they are likely to encounter. And, secondly, we would suggest that our results are useful in that they provide insights into the nature and magnitudes of the risks associated with different types of mortality position. In doing so, they also give a sense of the prospective basis risks associated with mortality hedge positions, and a good appreciation of the size of the basis risks is important if risk managers are to make the best possible use of mortality derivatives for hedging purposes. In short, the framework presented here gives life offices a means of measuring their mortality risks and this, in turn, should help them manage those risks more effectively than they have hitherto been able to do.

Acknowledgements

This work was carried out under the auspices of two research grants from the ESRC (RES-000-23-1036 and RES-00-27-0014) and the authors thank the ESRC for their support. The authors also thank an anonymous referee for helpful comments. However, the usual caveat applies.

Appendix A. Simulating the parameters of the model

This appendix shows how to simulate values of the μ and V parameters of the mortality risk model. Our original data consist of sample values of $\hat{\mu}$, $\hat{V}$ and the sample size n. For our data spanning 1961–2002, $\hat{\mu}$ and $\hat{V}$ are as given in (4a) and (4b) and $n = 41$.

Our first task is to simulate V from its posterior distribution, the Wishart($n-1, n^{-1}\hat{V}^{-1}$) distribution. To do so, we carry out the following steps:

- We simulate $(n-1)$ 2×1 i.i.d. vectors $\alpha_1, \ldots, \alpha_{n-1}$ from a multivariate normal distribution with mean vector 0 and covariance matrix $n^{-1}\hat{V}^{-1}$.
- We construct the 2×2 matrix $X = \sum_{i=1}^{n-1}\alpha_i\alpha_i^{\mathrm{T}}$.
- We then invert X to obtain our simulated positive-definite covariance matrix $V(=X^{-1})$.

Having obtained our simulated V matrix, we simulate μ from a MVN($\hat{\mu}, n^{-1}V$).

References

Acerbi, C., 2004. Coherent representations of subjective risk aversion. In: Szegö, G. (Ed.), Risk Measures for the 21st Century. Wiley, New York, pp. 147–207.

Benjamin, B., Pollard, J.H., 1993. The Analysis of Mortality and Other Actuarial Statistics, third ed. Institute of Actuaries, London.

Blake, D., Burrows, W., 2001. Survivor bonds: helping to hedge mortality risk. Journal of Risk and Insurance 68, 339–348.

Cairns, A.J.G., Blake, D., Dowd, K., 2006a. Pricing death: frameworks for the valuation and securitization of mortality risk. ASTIN Bulletin, in press.

Cairns, A.J.G., Blake, D., Dowd, K., 2006b. A two-factor model of stochastic mortality. Journal of Risk and Insurance, in press.

Dahl, M., 2004. Stochastic mortality in life insurance: market reserves and mortality-linked insurance products. Insurance: Mathematics and Economics 35, 113–136.

Dowd, K., 2005. Measuring Market Risk, second ed. Wiley, Chichester.

Dowd, K., Blake, D., Cairns, A.J.G., Dawson, P.E., 2006. Survivor swaps. Journal of Risk and Insurance, in press.

Lin, Y., Cox, S.H., 2004. Natural Hedging of Life and Annuity Mortality Risks. Georgia State University, mimeo.

Lin, Y., Cox, S.H., 2005a. Securitization of mortality risks in life annuities. Journal of Risk and Insurance 72, 227–252.

Lin, Y., Cox, S.H., 2005b. A mortality securitization model. Working Paper. Georgia State University.

Milevsky, M.A., Promislow, S.D., 2001. Mortality derivatives and the option to annuitize. Insurance: Mathematics and Economics 29, 299–318.

Perks, W., 1932. On some experiments in the graduation of mortality statistics. Journal of the Institute of Actuaries 63, 12–57.

Wang, S., 1996. Premium calculation by transforming the layer premium density. ASTIN Bulletin 26, 71–92.

Wang, S., 2000. A class of distortion operations for pricing financial and insurance risks. Journal of Risk and Insurance 67, 15–36.

Yang, S., 2001. Reserving, pricing and hedging for guaranteed annuity options. Ph.D. Thesis. Heriot-Watt University.