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Optimal Pension Asset Allocation Strategy when Terminal Utility is a Function of Replacement Ratio

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Optimal pension asset allocation strategy when terminal utility is a function of replacement ratio

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Abstract

This paper considers the optimal asset allocation problem for defined-contribution pension plan members whose terminal utility is a function of replacement ratio, i.e. the pension-to-final wage ratio. When three asset types are available for investment, the optimal portfolio composition, which is horizon dependent, includes investment in both riskless and risky assets. The investment in risky assets has three components to hedge wage risk, to speculate on risk premiums and to hedge for financial market risk respectively.

When the terminal utility is a power function, closed form solution is derived for the cases where there is no further contribution from wage incomes or there is no non-hedgeable wage risk. The horizon dependence of optimal pension portfolio is deterministic under assumptions of constant equity risk premium, constant interest rate volatility and constant stock return volatility. The short-sale of wage replicating portfolio also contributes to the horizon dependence of pension plan financial wealth (the sum of pension portfolio and the short-sold wage replicating portfolio), and the effect is stochastic due to the stochastic interest rate and stock return. Therefore, the optimal asset allocation strategy in terms of financial wealth is “stochastic lifestyling”.

For the cases where wage incomes cannot be hedged due to non-hedgeable wage risk, optimal asset proportions can be solved numerically by Monte Carlo simulation. The proportions invested in stocks and especially bonds are higher in early stages than those when wage replicating portfolio is used, hence more short-sale of cash assets. The optimal asset allocation derived by numerical simulation is also horizon dependent.

Keywords: Optimal asset allocation; Defined-contribution pension plan; Annuity; Power utility; Hamilton-Jacobi-Bellman equation.

1. Introduction

There are two types of pension plan, defined benefit (DB) plans and defined contribution (DC) plans, which are very different in their objectives. In a DB pension plan, where the benefits are fixed in advance according to final wages and years of service by the sponsor, the objective of fund sponsors is to maintain the fund in balance (and minimize the sponsor's contribution). In a DC pension plan, the objective is to maximize the expected terminal utility of plan members. A difference in the terminal utility function can lead to different optimal asset allocation strategy. Early studies by Samuelson (1969) and Merton (1969, 1971) on consumption and portfolio strategy show that, an individual with utility as power function of consumption will have a horizon independent optimal portfolio composition. However, when utility is power function of consumption above a subsistence level, the optimal portfolio composition becomes horizon dependent (Samuelson 1989). The function form of terminal utility and its argument(s) have a principal influence on the optimal asset allocation strategy.

Different variables have been proposed as the argument(s) of terminal utility function. Battocchio and Menoncin (2004) assume that terminal utility is an exponential function of real wealth (wealth-to-price index ratio). The optimal asset allocation is horizon dependent, with percentage of riskless assets increasing over time and percentages of stocks and bonds decreasing over time. Their study is explicitly in nominal terms, whereas most portfolio and pension studies are implicitly in real terms, i.e. wealth is real wealth. Two criticisms may be raised against using real wealth as the argument of terminal utility function: 1) it does not take the pre-retirement standard of living into account, and 2) it does not provide downsize protection. Boulier et al (2001) and Deelstra et al (2003) consider a terminal utility that is a power function of cash surplus over minimum guaranteed benefits, because there is a real need for a downside protection when considering the retirement savings products. The optimal pension asset allocation is also horizon dependent, with percentage of riskless assets increasing over time and percentages of stocks and bonds decreasing over time. Choosing wealth surplus over minimum guaranteed benefits provides the downsize protection, but still does not take the pre-retirement standard of living into consideration.

In many countries, the principal retirement income vehicle in DC pension plans is life annuities. A life annuity will guarantee fixed retirement payments for however long an individual lives, thus protecting the retiree from outliving her resources. It has been shown that annuitization of pension wealth is generally utility increasing, because it protects a retiree from outliving her resources on one hand and prevents under-consuming her resources on the other hand (Yaari 1965). Life annuities are bond-based investments with longevity insurance, so that the improvement in longevity will increase the cost for life annuity providers and drive up the price of life annuities, and falls in bond yields will also make annuities more expensive to pension plan members. The continuing improvement in longevity and low bond yields raise the question whether life annuity is of good value over a long retirement period given that equities generally outperform bonds over long horizons. For more risk-tolerant pension plan members, annuitization removes the potential benefits of higher returns of investment in stock market. For more risk-averse plan members, life annuities are more preferable because they provide guaranteed benefits. In certain sense, pension from a DB plan is a life annuity that takes the pre-retirement standard of living (final wage) into consideration.

To relate the terminal utility with pre-retirement standard of living and take into account the fact that the principal retirement income vehicle in DC pension plans is life annuities, Cairns et al (2006) assume that the terminal utility is a power function of replacement ratio (pension-to-final wage ratio). Their assumption on terminal utility and DB pension plans indicates implicitly the need to have pension income comparable to existing wages in DC plans and some role of habit formation (Spinnewyn 1981; Becker and Murphy 1988) in terminal utility. Cairns et al (2006) find that the optimal asset allocation strategy is stochastic lifestyling using three mutual funds made of N risky assets and one riskless asset, which are dominated by cash assets, bonds and equities respectively and termed “cash fund”, “bond fund” and “equity fund”. In their stochastic lifestyling strategy, for power utility the pension fund invests a constant proportion of wealth in “equity fund” and time-dependent proportions in “cash fund” and “bond fund” and switches from “cash fund” to “bond fund” as retirement approaches.

Cairns et al (2006) did not give a proof to their conclusion that the three mutual funds are dominated by cash, bonds and equities respectively. The dominance of cash in the “cash fund” can only be understood when the three mutual funds invest in both riskless and risky assets. The present paper extends the study of Cairns et al (2006) by investigating both analytically and numerically the optimal composition of those mutual funds and separating bonds explicitly from equities. For simplicity, I assume that the pension plan can invest in three assets, cash, bond and stock (Boulier et al 2001; Deelstra et al 2003; Battocchio and Menoncin 2004). Stochastic interest rates are assumed in the present study, as in most recent pension asset allocation strategy studies (Boulier et al 2001; Vigna and Haberman 2001; Haberman and Vigna 2002; Deelstra et al 2003; Battocchio and Menoncin 2004; Cairns et al 2006). Although deterministic wage incomes are often used in pension studies (Boulier et al 2001; Vigna and Haberman 2001; Haberman and Vigna 2002; Deelstra et al 2003), stochastic wage incomes are considered in the present study. I use the same assumption on wages as in Battocchio and Menoncin (2004) and Cairns et al (2006).

In the present study I find that the optimal asset allocation in risky assets consists of three components: (i) a preference free component to hedge wage risk, (ii) a speculative component, proportional to both the portfolio Sharpe ratio and the inverse of the relative risk aversion index, and (iii) a hedging component depending on the state variable parameters. The third component, which corresponds to the “bond” fund of Cairns et al (2006), contains only bonds, whose weight in the portfolio is horizon dependent. The other two components all contain both bonds and equities, whose weight in the portfolio is horizon independent. The overall optimal pension wealth composition is deterministically horizon dependent, with a shift between cash and bonds. When wage income is fully hedgeable and a wage replicating portfolio is short-sold, the pension plan financial wealth (the sum of pension wealth and the short-sold wage replicating portfolio) is stochastically horizon dependent, because of the stochastic interest rate and stock returns. When wage income is not hedgeable, the optimal portfolio composition is also horizon dependent and the optimal proportions invested in bonds and stocks are higher than those when a wage replicating portfolio is used.

This paper is organized as follows. Section 2 formulates the financial market, wage, pension wealth, and current replacement ratio growth models. Section 3 presents the optimization problem and the Hamilton-Jacobi-Bellman equation. Section 4 solves the optimal asset allocation problem for power utility analytically. Since optimization for power utility with non-hedgeable wage income risk cannot be solved in closed-form, I only solve the cases where pension contribution has stopped or wage risk is fully hedgeable. Section 5 solves the optimal asset allocation problem when wage income is not hedgeable by numerical simulation. Section 6 discusses and summarizes the results in this paper.

2. The model

In this section, the terminal utility, financial market structure, wage process and current replacement ratio process are presented.

2.1. Terminal utility

The same definition of terminal utility as in Cairns et al. (2006) is used here,

$$u\left[W(T)/X(T), r(T)\right] = u\left[X(T)/a(T, r(T))\right] = u\left[P(T)/Y(T)\right] = u\left[G(T)\right].$$

Where P(T) is the annual pension purchased at T,

$$P(T) = \frac{W(T)}{a(T, r(T))}$$

and

$$G(T) = \frac{P(T)}{Y(T)} = \frac{X(T)}{a(T, r(T))} \tag{1}$$

is the replacement ratio.

2.2. Market structure

The financial market, assumed similarly to those in Boulier et al (2001), Deelstra et al (2003) and Battocchio and Menoncin (2004), is frictionless and continuously open, with no arbitrage. The uncertainty in the financial market is described by two standard and independent Brownian motions $Z_r(t)$ and $Z_S(t)$ with $t \in [0, T]$, defined on a complete probability space (Ω, $\mathcal{F}$, P) where P is the real world probability. The filtration $\mathcal{F} = \mathcal{F}(t)$ $\forall t \in [0, T]$ generated by the Brownian motions can be interpreted as the information set available to the investor at time t.

The instantaneous risk-free rate of interest $r(t)$ follows an Ornstein-Uhlenbeck process (Vasicek model)

$$dr(t) = \alpha(\beta - r(t))dt + \sigma_r dZ_r(t),$$
$$r(0) = r_0. \qquad (2)$$

In equation (2), α and β are strictly positive constants, and σ_r is the volatility of interest rate. The process is mean-reverting, and the instantaneous drift $\alpha(\beta - r(t))$ represents a force pulling the process towards its long term mean β with magnitude proportional to the deviation of the process from the mean. The stochastic element $Z_r(t)$ causes the process to fluctuate around the level β (Vasicek, 1977).

The price of zero-coupon bonds for any date of maturity τ at time t, $B(t, \tau, r)$, with the instantaneous interest rate process described by equation (2), is governed by the diffusion equation (Vasicek 1977; Boulier et al 2001; Deelstra 2003)

$$\frac{dB(t,\tau,r)}{B(t,\tau,r)} = (r(t) + b(t,\tau)\sigma_r \xi)dt - b(t,\tau)\sigma_r dZ_r(t),$$
$$B(\tau,\tau) = 1,$$

where ξ, assumed to be constant, is the market price of interest rate risk, and

$$b(t,\tau) = \frac{1 - e^{-\alpha(\tau - t)}}{\alpha}.$$

There are three types of asset in the financial market for investment. The first is a riskless asset, which follows a price process governed by

$$dR(t) = R(t)r(t)dt,$$
$$R(0) = R_0. \qquad (3)$$

The riskless asset can be considered as a cash fund, i.e. a bank account paying the instantaneous interest rate $r(t)$ without any default risk. The value of units in the cash fund at t is then

$$R(t) = R(0)\exp\left[\int_0^t r(s)ds\right]. \qquad (4)$$

The second asset is a bond rolling over zero coupon bonds with constant maturity K. The price of the zero coupon bond with constant maturity K is denoted by $B_K(t, r)$ with

$$\frac{dB_K(t,r)}{B_K(t,r)} = [r(t) + b_K \sigma_r \xi]dt - b_K \sigma_r dZ_r(t), \qquad (5)$$

where

$$b_K = \frac{1-e^{-\alpha K}}{\alpha} \; .$$

The relationship between $B(t,\ \tau,\ r)$ and $B_K(t,r)$ through the riskless cash asset $R(t)$ (Boulier et al, 2001) is

$$\frac{dB(t,\tau,r)}{B(t,\tau,r)} = \left(1 - \frac{b(t,\tau)}{b_K}\right)\frac{dR(t)}{R(t)} + \frac{b(t,\tau)}{b_K}\frac{dB_K(t,r)}{B_K(t,r)} \, .$$

The above equation shows that the "rolling bond" can be obtained by a portfolio of one zero coupon bond and the cash asset, and similarly, other bonds can be obtained through a portfolio of the riskless asset and the "rolling bond".

The third asset, stock, has a process of the total return (the value of a single premium investment in the stock with reinvestment of dividend income) governed by stochastic differential equation (SDE)

$$dS(t) = S(t)\left[\mu_S(r,t)dt + v_{rS}\sigma_r dZ_r(t) + \sigma_S dZ_S(t)\right],$$

$$S(0) = S_0 \, , \qquad (6)$$

where

$$\mu_S(r,t) = r(t) + m_S \qquad (7)$$

is the instantaneous percentage change in stock price per unit time. In equations (6) and (7), m_S is the risk premium on the stock, σ_S the stock specific volatility, and v_{rS} a volatility scale factor measuring how the interest rate volatility affects the stock volatility.

The market as assumed above has a diffusion matrix given by

$$\Sigma \equiv \begin{bmatrix} -b_K\sigma_r & 0 \\ v_{rS}\sigma_r & \sigma_S \end{bmatrix}, \qquad (8)$$

and σ_r and σ_S are assumed to be different from zero and the diffusion matrix is invertible.

2.3. Wages

The plan member's wage, $Y(t)$, evolves according to the SDE

$$dY(t) = Y(t)\left[(\mu_Y(t) + r(t))dt + v_{rY}\sigma_r dZ_r(t) + v_{SY}\sigma_S dZ_S(t) + \sigma_Y dZ_Y(t)\right],$$

$$Y(0) = Y_0 \, , \qquad (9)$$

where $\mu_Y(t)$ is a deterministic function of time, age and other individual characteristics such as education and occupations; σ_Y is a constant and $Z_Y(t)$ a standard Brownian motion, independent of $Z_r(t)$ and $Z_S(t)$; and v_{rY} and v_{SY} are volatility scaling factors measuring how interest rate volatility and stock volatility affect wage volatility, respectively. The parameter σ_Y is a non-hedgeable volatility whose risk source does not belong to the set of the financial market risk sources. When σ_Y = 0, the market is complete. Otherwise the market is incomplete.

2.4. Replacement ratio process

The value of the plan member's pension fund is denoted by $W(t)$, and the proportions of fund wealth invested in the riskless asset, bonds and stock are denoted as $\theta_R(t)$, $\theta_B(t)$ and $\theta_S(t)$ respectively,

$$\theta_R(t) + \theta_B(t) + \theta_S(t) = 1, \tag{10}$$

The SDE governing the pension wealth is

$$\begin{aligned} dW(t) &= W(t)\left[(1-\theta_B-\theta_S)\frac{dR}{R} + \theta_B\frac{dB}{B} + \theta_S\frac{dS}{S}\right] + \pi Y(t)dt \\ &= [W(t)r + W(t)\theta_B b_K \sigma_r \xi + W(t)\theta_S m_S + \pi Y(t)]dt \\ &+ W(t)(-\theta_B b_K + \theta_S v_{rS})\sigma_r dZ_r \\ &+ W(t)\theta_S \sigma_S dZ_S \,. \end{aligned} \tag{11}$$

Using Ito's formula, the process of current wealth-to-wage ratio, X(t)=W(t)/Y(t) is governed by the SDE

$$dX(t) = \frac{1}{Y}dW - \frac{W}{Y^2}dY + \frac{W}{Y^3}(dY)^2 - \frac{1}{Y^2}(dWdY) \;. \tag{12}$$

The current replacement ratio (the ratio between the life annuity if annuitizing the pension wealth now and the current wage) is defined as

$$G(t) = \frac{P(t)}{Y(t)} = \frac{X(t)}{a(t,r(t))} \,. \tag{13}$$

Using Ito's formula gives the process governing the replacement ratio

$$dG(t) = \frac{1}{a(t,r)}dX - \frac{X}{a(t,r)^2}da(t,r) + \frac{X}{a(t,r)^3}\left[da(t,r)\right]^2 - \frac{1}{a(t,r)^2}\left[dXda(t,r)\right]. \tag{14}$$

The process governing $a(t,r(t))$, which is actually a function of r(t) only, can be found by using Ito's formula,

$$da(t,r) = \frac{\partial a(t,r)}{\partial r} dr + \frac{1}{2} \frac{\partial^2 a(t,r)}{\partial r^2} (dr)^2 . \qquad (15)$$

Following Cairns et al (2006), the above equation can be written as

$$\begin{aligned} da(t,r) &= a(t,r)\left[-d_a(r)dr(t) + \frac{1}{2} c_a(r)(dr(t))^2 \right] \\ &= a(t,r)\left[\left(\frac{1}{2} c_a(r)\sigma_r{}^2 - d_a(r)\alpha(\beta - r) \right) dt - d_a(r)\sigma_r dZ_r \right]. \end{aligned} \qquad (16)$$

In the second equality, the Vasicek model of interest rate is used

$$dr(t) = \alpha(\beta - r(t))dt + \sigma_r dZ_r(t) .$$

In the equation, $d_a(r)$ is the duration of the annuity function, $c_a(r)$ is its convexity

$$d_a(r) = -\frac{1}{a(t,r)} \frac{\partial a(t,r)}{\partial r},$$

$$c_a(r) = \frac{1}{a(t,r)} \frac{\partial^2 a(t,r)}{\partial r^2} . \qquad (17)$$

By substituting *W*, *dW*, *X*, $a(t,r(t))$, dX and $da(t,r)$, the SDE governing the replacement ratio process is:

$$dG(t) = (\theta' MG + u)dt + (\theta'\Gamma' G + \Lambda')dZ, \qquad (18)$$

where

$$\theta = [\theta_B \quad \theta_S]',$$

$$M = \begin{bmatrix} b_K \sigma_r \xi + (v_{rY} - d_a) b_K \sigma_r{}^2 \\ m_S - (v_{rY} - d_a) v_{rS} \sigma_r{}^2 - v_{SY} \sigma_S{}^2 \end{bmatrix},$$

$$u = \frac{\pi}{a} + G[-\mu_Y + (v_{rY}{}^2 - d_a v_{rY} - \frac{1}{2} c_a + d_a{}^2)\sigma_r{}^2 + v_{SY}\sigma_S{}^2 + \sigma_Y{}^2 + d_a \alpha(\beta - r)],$$

$$\Gamma' = \begin{bmatrix} -b_K \sigma_r & 0 & 0 \\ v_{rS} \sigma_r & \sigma_S & 0 \end{bmatrix},$$

$$\Lambda = [(d_a - v_{rY})\sigma_r \quad -v_{SY}\sigma_S \quad -\sigma_Y]',$$

$$Z = [Z_r \quad Z_S \quad Z_Y]' . \qquad (19)$$

The new diffusion matrix for the financial market is given by Γ. (Γ’Γ) is assumed to be invertible. The objective of the pension fund manager is to choose a portfolio strategy that maximizes the expected value of a terminal utility function. The terminal utility function is a function of the replacement ratio G(t).

3. The optimization problem and Hamilton-Jacobi-Bellman equation

The stochastic optimal control problem is

$$\max_{\theta} E\left[U(G(T),T)\right],$$

subject to

$$dw = \mu_w dt + \Omega' dZ \,, \tag{20}$$

where,

$$w \equiv \begin{bmatrix} r & Y \end{bmatrix}',$$

$$\mu_w \equiv \begin{bmatrix} \alpha(\beta - r) & Y(\mu_Y + r) \end{bmatrix}',$$

$$\Omega' \equiv \begin{bmatrix} \sigma_r & 0 & 0 \\ Y v_{rY}\sigma_r & Y v_{SY}\sigma_S & Y\sigma_Y \end{bmatrix}. \tag{21}$$

The corresponding Hamiltonian is

$$H(J) = J_t + \mu'_w \frac{\partial J}{\partial w} + (\theta' MG + u)\frac{\partial J}{\partial G} + \frac{1}{2} tr\left(\Omega'\Omega \frac{\partial^2 J}{\partial w^2}\right) + (\theta'\Gamma' G + \Lambda')\Omega \frac{\partial^2 J}{\partial w \partial G}$$
$$+ \frac{1}{2}(\theta'\Gamma'\Gamma\theta G^2 + 2\theta'\Gamma'\Lambda G + \Lambda'\Lambda)\frac{\partial^2 J}{\partial G^2}. \tag{22}$$

The system of the first order conditions on H with respect to θ is:

$$\frac{\partial H}{\partial \theta} = MG\frac{\partial J}{\partial G} + \Gamma'\Omega G \frac{\partial^2 J}{\partial w \partial G} + (\Gamma'\Gamma\theta G^2 + \Gamma'\Lambda G)\frac{\partial^2 J}{\partial G^2} = 0\,. \tag{23}$$

The optimal portfolio composition is:

$$\theta^* = -\frac{\Gamma^{-1}\Lambda}{G} - (\Gamma'\Gamma)^{-1} M \frac{J_G}{GJ_{GG}} - (\Gamma'\Gamma)^{-1}\Gamma'\Omega \frac{J_{wG}}{GJ_{GG}}\,. \tag{24}$$

The three terms (components) on the right hand side of equation (24) can be viewed as three funds and designated as θ_1*, θ_2* and θ_3* respectively, which are themselves vectors with two elements corresponding to certain proportions of investment in bonds and stock. The three components correspond to the three mutual funds in Cairns et al (2006). From this analysis, we have the following proposition.

Proposition 1: *The optimal asset allocation in the risky assets consists of three components: (i) a preference-free hedging component,* $\theta^*_1 = -\frac{\Gamma^{-1}\Lambda}{G}$*; (ii) a speculative component proportional to both the portfolio Sharpe ratio and the inverse of the Arrow-Pratt absolute risk aversion index* $\theta^*_2 = -(\Gamma'\Gamma)^{-1} M \frac{J_G}{GJ_{GG}}$ *; and (iii) a hedging component depending on the state variable parameters* $\theta^*_3 = -\Gamma^{-1}\Omega \frac{J_{wG}}{GJ_{GG}}$.

It can be seen that the first portfolio component minimizes the instantaneous variance of replacement ratio differentials, since

$$Var(dG) = (\theta'\Gamma'\Gamma + 2\theta'\Gamma'\Lambda + \Lambda'\Lambda)dt\,. \qquad (25)$$

The second component is to satisfy the risk appetite of plan members and the third component is to hedge the financial market risks.

4. Optimal asset allocation for power terminal utility

If the expected terminal utility is of the form $J(t, G(t), w) = \frac{1}{1-\gamma} g(t,w)^{\gamma} G^{1-\gamma}$ where γ is the relative risk aversion coefficient, the optimal allocation strategy can be solved only for scenarios where 1) there is no contribution from future wage incomes ($\pi = 0$), and 2) but there is no non-hedgeable wage risk ($\sigma_Y = 0$). When there are both pension contributions from wage incomes ($\pi > 0$) and a non-hedgeable wage risk ($\sigma_Y \neq 0$), there is no analytical solution for the optimal asset allocation problem (20). In the present study I will only work on the two scenarios with analytical solutions.

4.1. Optimal asset allocation without wage income contribution, π=0

Since $\pi = 0$, and the plan member no longer makes pension contributions, u in the SDE governing pension ratio process can be rewritten as uG

$$u = -\mu_Y + ({v_{rY}}^2 - d_a v_{rY} - \frac{1}{2}c_a + {d_a}^2){\sigma_r}^2 + v_{SY}{\sigma_S}^2 + {\sigma_Y}^2 + d_a\alpha(\beta - r)\,.$$

(26)

Substituting the derivatives of the expected terminal power utility function ($J(t,G(t),w)=\frac{1}{1-\gamma}g(t,w)^{\gamma}G^{1-\gamma}$) and the optimal proportion composition of pension fund investment θ* into the above HJB equation and simplifying give (for details of derivation see Appendix)

$$g_t+\left[\mu_w'+\frac{1-\gamma}{\gamma}M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega\right]g_w+\frac{1}{2}tr(\Omega'\Omega g_{ww})$$
$$+\left[\frac{1-\gamma}{\gamma}u-\frac{1-\gamma}{\gamma}M'(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1-\gamma}{2(-\gamma)^2}M'(\Gamma'\Gamma)^{-1}M\right]g=0. \tag{27}$$

By the Feynman-Kac formula (Øksendal 2000; Duffie 2001), there exists a probability measure Q(γ) such that

$$g(t,w(t))=E_{Q(\gamma)}[g(T,\tilde{w}(T))D(t,T)\,|\,F_t]. \tag{28}$$

where $\tilde{w}(s)$ is governed by the SDE

$$d\tilde{w}(s)=\tilde{\mu}_w(\tilde{w}(s))ds+\Omega(\tilde{w}(s),s)'dZ\ ,$$
$$\tilde{\mu}_w(\tilde{w}(s))=\mu_w+\frac{1-\gamma}{\gamma}M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega\,,$$
$$\tilde{w}(t)=w(t)\,, \tag{29}$$

and

$$D(t,T)=\exp\left[\int_t^T\varphi(s)ds\right], \tag{30}$$

where

$$\varphi(s)=\frac{1-\gamma}{\gamma}u-\frac{1-\gamma}{\gamma}M'(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1-\gamma}{2(-\gamma)^2}M'(\Gamma'\Gamma)^{-1}M\ . \tag{31}$$

Since the terminal utility U(G(T)) depends only on the replacement ratio,

$$g(T,w(T))=1.$$

The optimal pension composition is

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda + (\Gamma'\Gamma)^{-1}M\frac{1}{\gamma} - (\Gamma'\Gamma)^{-1}\Gamma'\Omega\int_t^T \frac{\partial}{\partial w_t}E_t[\varphi(s)]ds\,. \qquad (32)$$

Since only u explicitly depends on the state variables,

$$\theta^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda + (\Gamma'\Gamma)^{-1}M\frac{1}{\gamma} + \frac{1-\gamma}{\gamma}(\Gamma'\Gamma)^{-1}\Gamma'\Omega\int_t^T \frac{\partial}{\partial w_t}E_t[u]ds\,. \qquad (33)$$

The first item in the right hand side of the above equation is

$$\theta_1^* = -(\Gamma'\Gamma)^{-1}\Gamma'\Lambda = \frac{1}{b_K}\begin{bmatrix} v_{rS}v_{SY} - (v_{rY} - d_a) \\ b_K v_{SY} \end{bmatrix}. \qquad (34)$$

This preference free component is to hedge wage risk, corresponding to the “cash” fund in Cairns et al (2006). Since this term represents a proportion in risky assets, it does not contain any cash asset and therefore cannot be dominated by cash. In Cairns et al (2006), the investment in riskless assets is split into three parts and added to the three mutual funds, hence the “cash” fund contains cash assets and can be dominated by cash assets.

The second item is

$$\theta_2^* = \frac{(\Gamma'\Gamma)^{-1}M}{\gamma} = \frac{1}{b_K\sigma_r\sigma_S^2\gamma}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix}. \qquad (35)$$

where

$$\phi_1 = \xi(v_{rS}^2\sigma_r^2 + \sigma_S^2) + (v_{rY} - d_a)\sigma_r\sigma_S^2 + v_{rS}\sigma_r m_S - v_{rS}v_{SY}\sigma_r\sigma_S^2,$$

$$\phi_2 = b_K(\xi v_{rS}\sigma_r^2 + m_S\sigma_r - v_{SY}\sigma_r\sigma_S^2)\,.$$

This second item (speculative component) contains both bonds and equities, which corresponds to the “equity fund” of Cairns et al (2006). The present result seems not so consistent with the conclusion of Cairns et al (2006) that the “equity” fund is dominated by equities. Using the usual assumptions on market parameters in Table 1, I calculate the proportions of bonds and stocks in the speculative component. As shown in Fig.1, for the market parameters assumed, the “equity” fund is dominated by bonds instead of stocks for a wide range of relative risk aversion coefficient γ.

Table 1 Parameters used in numerical simulation

Interest rate	Value
Mean reversion, α,	0.2
Mean rate, β	0.05
Volatility, σ_r	0.02
Initial rate, r_0	0.05
Fixed maturity bond	
Maturity, K	20 years
Market price of risk, ξ	0.15
Stock	
Risk Premium, m_S	0.06
Stock own volatility, σ_S	0.19
Interest volatility scale factor, v_{rS}	1
Wage	
Wage premium, μ_Y	0.01
Non-hedgeable volatility, σ_Y	0.01
Interest volatility scale factor, v_{rY}	0.7
Stock volatility scale factor, v_{SY}	0.9
Initial wage, Y_0	10k
Contribution rate, π	10%
Length of pension plan, T	45

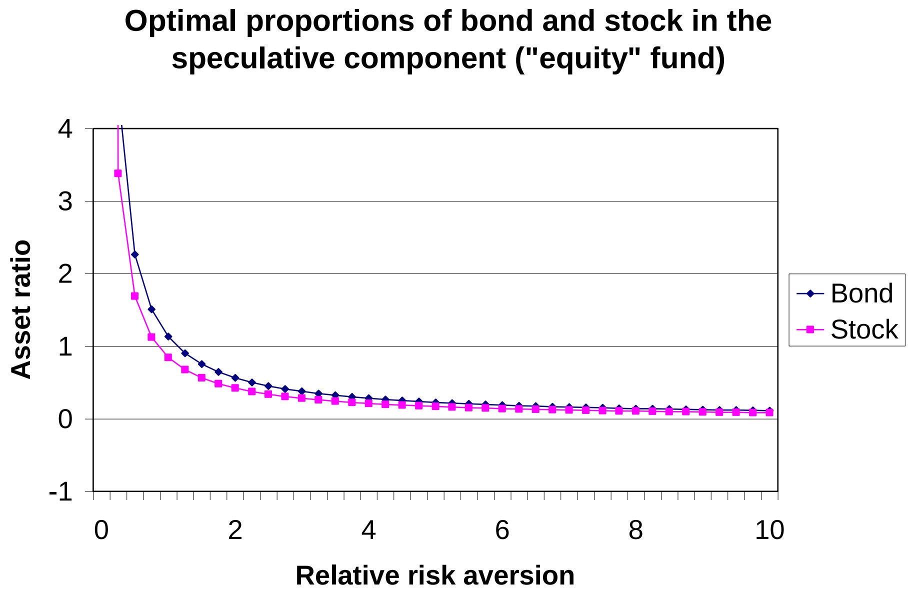

Fig.1 The relationship between the optimal proportions of bond and stock in the speculative component and the relative risk aversion coefficient γ. Parameters in Table 1 are used for calculation.

It is necessary to find out the modified process of r for computing the third term in the above equation. The matrix product

$$M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega = \begin{bmatrix} -\xi\sigma_r + (d_a - v_{ry})\sigma_r^{\ 2} \\ -Yv_{rY}[\xi\sigma_r + (v_{ry} - d_a)\sigma_r^{\ 2}] + Yv_{SY}(v_{rS}\xi\sigma_r + m_S - v_{SY}\sigma_S^{\ 2}) \end{bmatrix}. \tag{36}$$

The modified processes of state variables are

$$\begin{bmatrix} d\tilde{r} \\ d\tilde{Y} \end{bmatrix} = \begin{bmatrix} \alpha(\beta - \tilde{r}) + \dfrac{(1-\gamma)[(d_a - v_{ry})\sigma_r^{\ 2} - \sigma_r\xi]}{\gamma}\sigma_r^{\ 2} \\ \tilde{Y}\left((\mu_Y + \tilde{r}) + \dfrac{(1-\gamma)[(v_{rY} - d_a)v_{rY}\sigma_r^{\ 2} + v_{SY}^{\ 2}\sigma_S^{\ 2} + (v_{rY} - v_{rS}v_{SY})\xi\sigma_r - v_{SY}m_S]}{\gamma}\right) \end{bmatrix} dt$$

$$+ \begin{bmatrix} \sigma_r & 0 & 0 \\ \tilde{Y}v_{rY}\sigma_r & \tilde{Y}v_{SY}\sigma_S & \tilde{Y}\sigma_Y \end{bmatrix} \begin{bmatrix} dZ_r \\ dZ_S \\ dZ_Y \end{bmatrix}. \tag{37}$$

The solutions of the above processes, for $s \geq t$, are

$$\tilde{r}(s) = \tilde{r}(t)e^{\alpha(t-s)} + \frac{\alpha\beta + \frac{(1-\gamma)[(d_a - v_{ry})\sigma_r{}^2 - \sigma_r\xi]}{\gamma}\sigma_r{}^2}{\alpha}\left(1 - e^{\alpha(t-s)}\right) \tag{38}$$

$$+ \sigma_r \int_t^s e^{-\alpha(s-\tau)} dZ_r(\tau).$$

$$\tilde{Y}(s) = \tilde{Y}(t)\exp\Big(\int_t^s \tilde{r}(\tau)d\tau + \Big(\mu_Y$$
$$+ \frac{(1-\gamma)[(v_{rY} - d_a)v_{rY}\sigma_r{}^2 + v_{SY}{}^2\sigma_S{}^2 + (v_{rY} - v_{rS}v_{SY})\xi\sigma_r - v_{SY}m_S]}{\gamma}$$
$$+ v_{SY}{}^2\sigma_S{}^2 + \sigma_Y{}^2 - \frac{1}{2}v_{rY}{}^2\sigma_r{}^2 - \frac{1}{2}v_{sY}{}^2\sigma_S{}^2 - \frac{1}{2}\sigma_Y{}^2\Big)(s-t)$$
$$+ v_{rY}\sigma_r(Z_r(s) - Z_r(t)) + v_{sY}\sigma_S(Z_S(s) - Z_S(t)) + \sigma_Y(Z_Y(s) - Z_Y(t))\Big). \tag{39}$$

The expected value of the modified interest rate process at time t is

$$E_t[\tilde{r}(s)] = \tilde{r}(t)e^{\alpha(t-s)} + \frac{\alpha\beta + \frac{(1-\gamma)[(d_a - v_{ry})\sigma_r{}^2 - \sigma_r\xi]}{\gamma}\sigma_r{}^2}{\alpha}\left(1 - e^{\alpha(t-s)}\right),$$

$$\tilde{r}(t) = r(t) \text{ (the boundary condition) }. \tag{40}$$

The integral in the third term is

$$\int_t^T \frac{\partial}{\partial w_t} E_t[u]ds = \begin{bmatrix} \int_t^T \frac{\partial E_t[u]}{\partial r(t)} ds \\ \int_t^T \frac{\partial E_t[u]}{\partial Y(t)} ds \end{bmatrix} = \begin{bmatrix} \int_t^T - d_a \alpha e^{\alpha(t-s)} ds \\ 0 \end{bmatrix} = \begin{bmatrix} d_a\left(e^{\alpha(t-T)} - 1\right) \\ 0 \end{bmatrix}. \tag{41}$$

The third item

$$\theta_3{}^* = \frac{1-\gamma}{\gamma}(\Gamma'\Gamma)^{-1}\Gamma'\Omega\int_t^T \frac{\partial}{\partial w_t} E_t[u]ds$$
$$= \frac{1-\gamma}{\gamma b_K}\begin{bmatrix} -1 & -Y(v_{rY} - v_{rS}v_{SY}) \\ 0 & Yb_K v_{SY} \end{bmatrix}\begin{bmatrix} d_a\left(e^{\alpha(t-T)} - 1\right) \\ 0 \end{bmatrix} \tag{42}$$
$$= \frac{1-\gamma}{\gamma b_K}\begin{bmatrix} -d_a\left(e^{\alpha(t-T)} - 1\right) \\ 0 \end{bmatrix}.$$

This state variable dependent hedging component contains only bonds and it is horizon-dependent. The optimal proportions of pension wealth invested in bonds and equities are

$$
\begin{bmatrix} \theta_B * \\ \theta_S * \end{bmatrix} = \frac{1}{b_K}\begin{bmatrix} d_a - v_{rY} + v_{rS} v_{SY} \\ b_K v_{SY} \end{bmatrix}
+ \frac{1}{\gamma b_K \sigma_r \sigma_S{}^2}\begin{bmatrix} v_{rS}{}^2 \sigma_r{}^2 \xi + \sigma_S{}^2 \xi + (v_{rY} - d_a)\sigma_r \sigma_S{}^2 + v_{rS} m_S \sigma_r - v_{rS} v_{SY} \sigma_r \sigma_S{}^2 \\ b_K (v_{rS} \sigma_r{}^2 \xi + m_S - v_{SY}{}^2 \sigma_r \sigma_S{}^2) \end{bmatrix}
+ \frac{1-\gamma}{\gamma b_K}\begin{bmatrix} - d_a \left(e^{\alpha(t-T)} - 1\right) \\ 0 \end{bmatrix}. \tag{43}
$$

The optimal proportion of pension wealth invested in risk-free assets can be calculated by using

$$\theta_R(t)* = 1 - \theta_B(t)* - \theta_S(t)*. \tag{44}$$

From the above results, it is clear that optimal asset allocation is horizon-dependent when terminal utility is a function of the replacement ratio. The horizon dependent change is a shift between cash and bond funds, whereas the proportion in the stock is constant. When the relative risk aversion $\gamma>1$, there is shift from cash to bond over time. When the relative risk aversion $\gamma<1$, there is shift from bond to cash over time. When the relative risk aversion $\gamma=1$, there is no shift between bond to cash over time.

The wage and current pension ratio process is simulated with parameters in Table1 by a numerical method (Higham 2001). I assume that $a(t, r(t))$ is of the form $\exp[d_0 - d_1 r(t)]$ with $d_0 = 3$ and $d_1 = 3.5$, as Cairns et al (2006). This assumption of the annuity rate implies $d_a = 3.5$ and $c_a = 12.25$. Because of the assumption that there is no further contribution, the initial replacement ratio has been assumed to be 0.05. Fig. 2 shows the relationship between relative risk aversion and the evolution of the replacement ratio for $\gamma \geq 1$. The larger the relative risk aversion, the smaller the expected final replacement ratio, because the less risk averse individual will short-sell more cash asset to hold more assets in bond and stock.

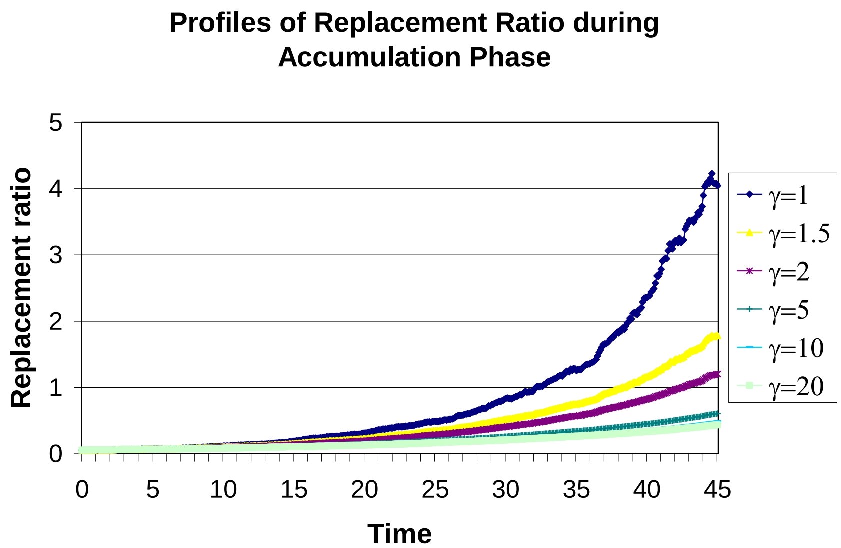

Fig.2 The evolution of current replacement ratio during the accumulation phase for different values of relative risk aversion, no contribution. Results are from 100 simulations for each curve.

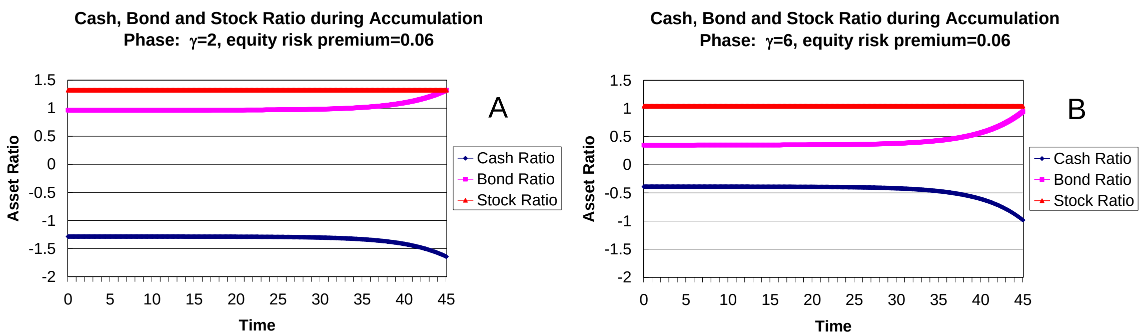

Fig.3 The evolution of optimal proportions of pension wealth invested in cash, bond and stock d during the accumulation phase for two different values of relative risk aversion, γ. A. γ =2. B. γ =6.

When γ<1, the terminal replacement ratio increases with γ initially. After reaching a maximum, the terminal replacement ratio decreases as γ increases. With parameters in

Table 1, γ for the maximum of expected terminal replacement ratio is around 0.6, when the short-sale of cash asset is the highest. Fig.3 shows how the optimal proportions of pension wealth invested in cash, bond and stock evolve during the accumulation phase of the pension plan. The proportion invested in stock is constant over lifetime of the pension plan. For the more risk averse plan members (γ=6), the proportions in stock and bond are smaller than more risk tolerant plan members (γ=2), so is the short-sale of cash assets. For γ>1, the shift from cash to bond over time is actually an increase in the amount of cash borrowed in order to buy more bonds.

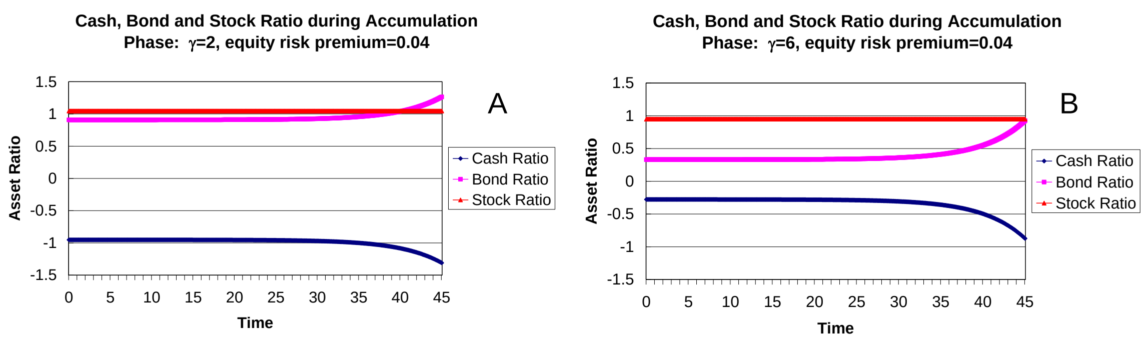

Fig.4 The evolution of optimal proportions of pension wealth invested in cash, bond and stock d during the accumulation phase for two different values of relative risk aversion, γ, with the risk premium m_S=0.04. A. γ =2. B. γ =6.

The parameters used in the simulations, like volatilities, σ_r and $\sigma_{S,}$ have a profound influence on the optimal proportions of the three assets. The proportion of wealth invested in the stock decreases in the stock volatility σ_S and increases in the stock risk premium m_S. The proportion invested in the bond decreases in interest rate volatility σ_r. The proportion invested in cash assets increases in σ_S and decreases in interest rate volatility σ_r. Those parameters commonly used in pension and portfolio studies tend to indicate that equities are not so risky relative to their expected returns. The risk premium appears to more than compensate the extra risk. The equity risk premium of 0.06 in Table 1 is an estimate based on historical stock return data in the United States, and it may not be as high in future as in the past (Blanchard 1993; Campbell and Shiller 2001; Fama and French 2002; Jagannathan, McGrattan and Scherbina 2001). Fig.4 shows the optimal asset allocation with an equity risk premium of 0.04, which is fairly common choice in recent literature (Fama and French 2002; Campbell and Viceira 2002; Gomes and Michaelides 2005). The proportion invested in stocks is much lower than that with an equity risk premium of 0.06, while the proportion in bonds only marginally lower.

4.2. Optimal asset allocation with hedgeable wage income

When the market is complete and wage incomes are fully hedgeable ($\sigma_Y = 0$), u, Γ, Λ and Z in equation (18), become

$$u = \frac{\pi}{a} + G[-\mu_Y + (v_{rY}{}^2 - d_a v_{rY} - \frac{1}{2}c_a + d_a{}^2)\sigma_r{}^2 + v_{SY}\sigma_S{}^2 + d_a\alpha(\beta - r)],$$

$$\Gamma' = \begin{bmatrix} -b_K\sigma_r & 0 \\ v_{rS}\sigma_r & \sigma_S \end{bmatrix},$$

$$\Lambda = \left[(d_a - v_{rY})\sigma_r \quad -v_{SY}\sigma_S\right]',$$

$$Z = \left[Z_r \quad Z_S\right]'. \tag{45}$$

The diffusing term of the state variables in equation (20) becomes

$$\underset{2\times 2}{\Omega'} \equiv \begin{bmatrix} \sigma_r & 0 \\ Yv_{rY}\sigma_r & Yv_{SY}\sigma_S \end{bmatrix}. \tag{46}$$

To borrow against future wage income it is necessary to calculate the market value at time *t* for future premiums payable between *t* and *T*. Let Q be the risk-neutral pricing measure and $\tilde{Z}_r(t)$ and $\tilde{Z}_S(t)$ independent standard Q-Brownian motions (Cairns et al 2006), the wage process under Q is

$$dY(t) = Y(t)\left[(\mu_Y(t) + r(t) - \xi_r v_{rY}\sigma_r - \xi_S v_{SY}\sigma_S)dt + v_{rY}\sigma_r d\tilde{Z}_r(t) + v_{SY}\sigma_S d\tilde{Z}_S(t)\right], \tag{47}$$

which implies that

$$Y(\tau) = Y(t)\exp\Big\{\int_t^\tau [\mu_Y(s) + r(s)]ds - (\xi_r v_{rY}\sigma_r + \xi_S v_{SY}\sigma_S + \frac{1}{2}v_{rY}{}^2\sigma_r{}^2 + \frac{1}{2}v_{SY}{}^2\sigma_S{}^2)(\tau - t)$$
$$+ v_{rY}\sigma_r[\tilde{Z}_r(\tau) - \tilde{Z}_r(t)] + v_{SY}\sigma_S[\tilde{Z}_S(\tau) - \tilde{Z}_S(t)]\Big\}.$$

(48)

Here ξ_r is a measure of how interest/bond volatility will affect wage, and ξ_S is a scale factor measuring how stock price volatility affects wages. The market value at time *t* for future premiums payable between t and T is then

$$E_Q\left[\int_t^T \exp\left\{-\int_t^\tau r(s)ds\right\}\pi Y(\tau)d\tau \mid F_t\right]$$
$$= \pi E_Q\left[\int_t^T Y(t)\exp\left\{\int_t^\tau \mu_Y(s)ds - (\xi_r v_{rY}\sigma_r + \xi_S v_{SY}\sigma_S + \frac{1}{2}v_{rY}{}^2\sigma_r{}^2 + \frac{1}{2}v_{SY}{}^2\sigma_S{}^2)(\tau - t)\right.\right.$$
$$\left.\left. + v_{rY}\sigma_r[\tilde{Z}_r(\tau) - \tilde{Z}_r(t)] + v_{SY}\sigma_S[\tilde{Z}_S(\tau) - \tilde{Z}_S(t)]\right\}d\tau \mid F_t\right]$$
$$= \pi Y(t)\int_t^T \exp\left\{\int_t^\tau \mu_Y(s)ds - (\xi_r v_{rY}\sigma_r + \xi_S v_{SY}\sigma_S)(\tau - t)\right\}d\tau$$
$$= \pi Y(t) f(t).$$

(49)

The pension plan can have an additional wealth of $\pi Y(t) f(t)$ by short-selling a replicating portfolio of value $-\pi Y(t) f(t)$, which will be paid off exactly by future contributions from wage incomes. The total pension wealth enhanced with the present market value of future contributions is $W(t) + \pi Y(t) f(t)$. The optimal composition of pension fund will be the same as in the case of no wage income contribution, but the optimal terminal utility function will have the form

$$J(t, G, w) = \frac{1}{1-\gamma} g(t, w)^\gamma \left(G + \pi \frac{f(t)}{a(t, r(t))}\right)^{1-\gamma}. \qquad (50)$$

The optimal strategy is to hold $-\pi Y(t) f(t)$ in the replicating portfolio and invest the $W(t) + \pi Y(t) f(t)$ in the optimal composition of pension fund wealth. Such a treatment of hedgeable wage income risk is often applied in portfolio and pension studies (Deestra et al 2003; Cairns et al 2006). The composition of the replicating portfolio can be written in vector form

$$\theta^R = \begin{bmatrix} \theta_B{}^R \\ \theta_S{}^R \\ \theta_R{}^R \end{bmatrix} = \begin{bmatrix} \dfrac{v_{rS} v_{SY} - v_{rY}}{b_K} \\ v_{SY} \\ 1 - v_{SY} - \dfrac{v_{rS} v_{SY} - v_{rY}}{b_K} \end{bmatrix}. \quad (51)$$

In the above equation, the superscript R indicates replicating portfolio.

The sum $\widetilde{W}(t) = W(t) + \pi Y(t) f(t)$ can be denoted as the augmented pension wealth and $\widetilde{G}(t) = G(t) + \pi \dfrac{f(t)}{a(t,r)}$ augmented wealth-to-wage ratio. The matrix representation of the replacement ratio SDE and the HJB equation for the augmented replacement ratio is the same as that when there is no further income contribution, although the parameters u, Γ, Λ, Z and Ω in equations (18) and (20) are replaced by those defined in equations (45) and (46). The optimal portfolio composition when expressed in matrix form is the same as that without further contributions from wage incomes in the preceding subsection.

Substituting Γ, Λ, Z and Ω in equations (45) and (46) into the optimal solution, equation (27) or (28), the optimal proportions of pension wealth invested in bonds and equities are

$$\begin{bmatrix} \theta_B * \\ \theta_S * \end{bmatrix} = \frac{1}{b_K} \begin{bmatrix} d_a - v_{rY} + v_{rS} v_{SY} \\ b_K v_{SY} \end{bmatrix} + \frac{1-\gamma}{\gamma b_K} \begin{bmatrix} -d_a \left(e^{\alpha(t-T)} - 1\right) \\ 0 \end{bmatrix}$$
$$+ \frac{1}{\gamma b_K \sigma_r \sigma_S{}^2} \begin{bmatrix} v_{rS}{}^2 \sigma_r{}^2 \xi + \sigma_S{}^2 \xi + (v_{rY} - d_a) \sigma_r \sigma_S{}^2 + v_{rS} m_S \sigma_r - v_{rS} v_{SY} \sigma_r \sigma_S{}^2 \\ b_K (v_{rS} \sigma_r{}^2 \xi + m_S - v_{SY}{}^2 \sigma_r \sigma_S{}^2) \end{bmatrix}. \quad (52)$$

The optimal proportion of pension wealth invested in risk-free assets can be calculated by using

$$\theta_R(t)* = 1 - \theta_B(t)* - \theta_S(t)*. \quad (53)$$

From the above results, it is clear that the optimal proportions of pension wealth invested in the three asset categories are horizon-dependent. The horizon dependent change, like that with non-hedgeable wage risk and no further pension contribution, only happens in the cash and bond funds. The proportion invested in the stock does not change.

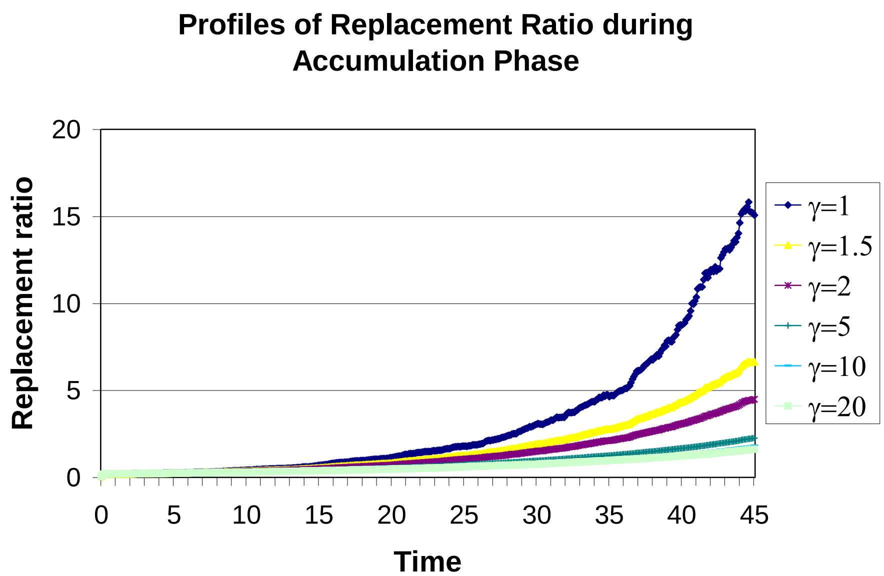

Fig.5 The evolution of current replacement ratio during the accumulation phase for different values of relative risk aversion, with fully hedgeable wage income. Results are from 100 simulations for each curve.

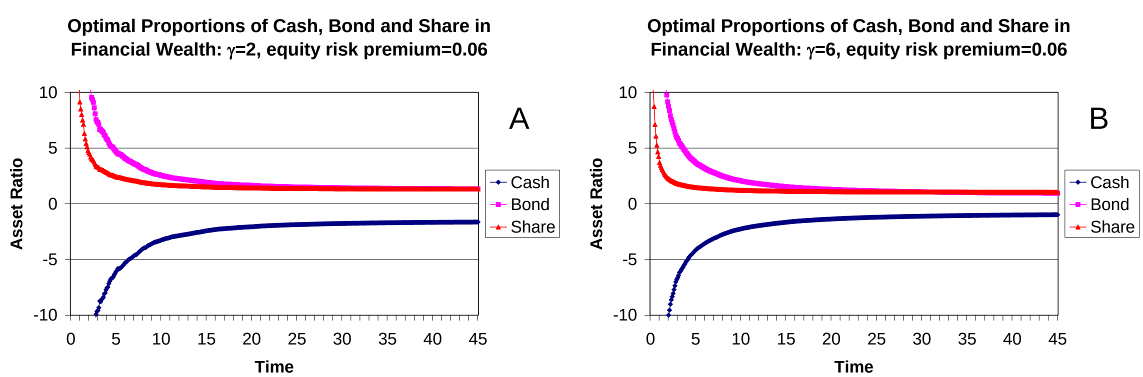

Fig.6 The evolution of optimal proportions of financial wealth (the sum of pension wealth and the short-sold wage replicating portfolio) invested in cash, bond and stock during the accumulation phase for two different values of relative risk aversion, γ. A. γ =2. B. γ =6.

Using the same set of parameters as those in Table 1, I have simulated the current pension ratio process for hedgeable wage income with a contribution rate of 10% (Fig. 5). The patterns of results are essentially the same as that of a lump sum invested at the beginning with non-hedgeable wage risk and no further pension contribution. With the short-sale of wage-replicating portfolio, the pension plan in fact consists of two portfolios. One is the pension portfolio, and the other the short-sold wage replicating portfolio which is to be paid off by future wage contributions. The sum of the two is the financial wealth (in hand) of the pension plan. Since the value of the wage replicating portfolio will decrease gradually and stochastically to zero by the time of retirement, it adds further horizon dependence of the optimal portfolio composition. Fig.6 shows the optimal proportions invested in the financial wealth over lifetime of the pension plan.

5. Optimal asset allocation when wage income is not hedgeable

In section 4, the wage income is hedgeable such that future pension contributions can be used in pension portfolio investment by short-selling a portfolio that replicates the wage income. If the wage income is not hedgeable, there is no analytical solution for the optimal asset allocation problem with power utility. With the market parameters in Table 1, the optimal proportions invested in cash, bond and stock are derived by numerical simulation. Because deriving the optimal asset allocation strategy for a continuous-time process spanning 45 years by Monte Carlo simulation needs considerable computing power, the solution is approximated by dividing the 45 years into 10 periods starting at the beginning of year 1, 6, 11, 16, 21, 26, 31, 36, 41 and 45 respectively and assuming that the asset allocation strategy does not change within each period. With each period, the asset return and the growth of pension wealth are simulated as continuous-time processes. This simplification greatly reduces the computing complexity of the problem, and the results still provide some useful insight on the optimal asset allocation strategy when wage incomes are not hedgeable.

The simulation is performed by using the same numerical method in Section 4 (Higham 2001) for stochastic differential equation. An allocation grid of proportions in bond and stock in each period is used to generate the terminal wealth and calculate the terminal utility. One hundred simulations are carried out for each sequence of combinations over the 10 periods. The expected terminal utility is calculated as the average of terminal utilities from the 100 simulations. The sequence of combinations over the 10 periods that gives the highest expected terminal utility is the (approximated) optimal asset allocation strategy.

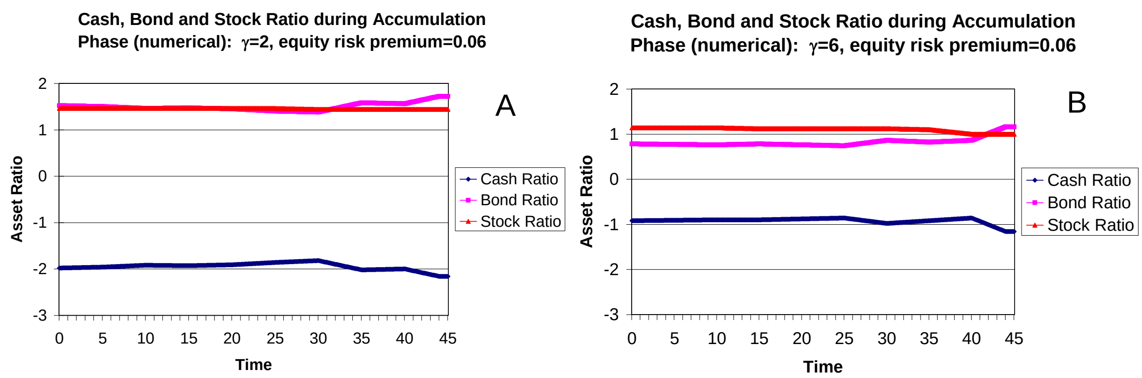

Fig.7 The evolution of optimal proportions of pension wealth invested in cash, bond and stock during the accumulation phase when the wage income is not hedgeable, for two different values of relative risk aversion, γ. A. γ =2. B. γ =6. Results are from 100 simulations for each curve.

The results are shown in Fig.7. The optimal asset allocation is still horizon dependent. The proportions invested in bonds and stocks are higher in early stages than those when wage replicating portfolio is used, hence more short-sale of cash assets. The increase in the proportion of risky assets is mainly in bonds. The increase in stock is small. There are two opposing effects in the evolution of optimal proportion in bonds. The effect of non-hedgeable wage decreases the optimal proportion in bond (and stock) as retirement approaches; while the optimal proportion in bond without wage income contribution increases as retirement approaches. Since neither effect is linear, the change in the optimal proportion over time seems to have a pattern of decrease, increase, decrease and then increase.

6. Conclusion

The optimal asset allocation problem during accumulation phase for terminal utility that is a function of replacement ratio is solved analytically and numerically in this paper. When terminal utility is a function of replacement ratio, the optimal portfolio composition is horizon-dependent. The proportion invested in equities is constant when wage income is hedgeable. There is a switch between cash and stock depends as the retirement approaches. When relative risk aversion $\gamma>1$, the switch is from cash to bonds; when $\gamma<1$, the switch is from bonds to cash. The proportion invested in the speculative component, which corresponds to the "equity fund" of Cairns et al (2006), is constant, but the speculative component contains both bonds and equities.

The horizon dependence of optimal pension portfolio is deterministic under assumptions of constant equity risk premium, constant interest rate volatility and constant stock return volatility. The short-sale of wage replicating portfolio also contributes to the horizon dependence of pension plan financial wealth (the sum of pension portfolio and the short-sold wage replicating portfolio), and the effect is stochastic due to the stochastic interest rate and stock return. Therefore, the optimal asset allocation strategy in terms of financial wealth is "stochastic lifestyling".

When wage incomes cannot be hedged due to non-hedgeable wage risk, optimal asset proportions can be solved numerically by Monte Carlo simulation. The optimal asset allocation is also horizon dependent. The proportions invested in stocks and especially in bonds are higher in early stages than those when wage replicating portfolio is used, hence more short-sale of cash assets.

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Appendix

Substituting the derivatives of the expected terminal power utility function ($J(t,G(t),w)=\frac{1}{1-\gamma}g(t,w)^{\gamma}G^{1-\gamma}$) into the above HJB equation gives

$$\frac{\gamma}{1-\gamma}g^{\gamma-1}g_tG^{1-\gamma}+(\theta' M+u)g^{\gamma}G^{1-\gamma}+\mu_w'\frac{\gamma}{1-\gamma}g^{\gamma-1}g_wG^{1-\gamma}$$
$$+\frac{1}{2}tr\left\{\Omega'\Omega\left[(-\gamma)g^{\gamma-2}g_w{}^2G^{1-\gamma}+\frac{\gamma}{1-\gamma}g^{\gamma-1}g_{ww}G^{1-\gamma}\right]\right\}+\theta'\Gamma'\Omega\gamma g^{\gamma-1}g_wG^{1-\gamma}$$
$$+\frac{1}{2}\left[\theta'\Gamma'\Gamma\theta+2\theta'\Gamma'\Lambda+\Lambda'\Lambda\right](-\gamma)g^{\gamma}G^{1-\gamma}=0. \tag{A-1}$$

Substituting the optimal proportion composition of pension fund investment θ* gives

$$\frac{\gamma}{1-\gamma}g^{\gamma-1}g_tG^{1-\gamma}+\left[\left((\Gamma'\Gamma)^{-1}\Gamma'\Omega g^{-1}g_w-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1}{-\gamma}(\Gamma'\Gamma)^{-1}M\right)'M+u\right]g^{\gamma}G^{1-\gamma}$$
$$+\mu_w'\frac{\gamma}{1-\gamma}g^{\gamma-1}g_wG^{1-\gamma}+\frac{1}{2}tr\left\{\Omega'\Omega\left[(-\gamma)g^{\gamma-2}g_w{}^2G^{1-\gamma}+\frac{\gamma}{1-\gamma}g^{\gamma-1}g_{ww}G^{1-\gamma}\right]\right\}$$
$$+\left((\Gamma'\Gamma)^{-1}\Gamma'\Omega g^{-1}g_w-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1}{-\gamma}(\Gamma'\Gamma)^{-1}M\right)'\Gamma'\Omega\gamma g^{\gamma-1}g_wG^{1-\gamma}$$
$$+\frac{1}{2}\left[\begin{array}{l}\left((\Gamma'\Gamma)^{-1}\Gamma'\Omega g^{-1}g_w-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1}{-\gamma}(\Gamma'\Gamma)^{-1}M\right)'\Gamma'\Gamma\\\left((\Gamma'\Gamma)^{-1}\Gamma'\Omega g^{-1}g_w-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1}{-\gamma}(\Gamma'\Gamma)^{-1}M\right)\\+2\left((\Gamma'\Gamma)^{-1}\Gamma'\Omega g^{-1}g_w-(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1}{-\gamma}(\Gamma'\Gamma)^{-1}M\right)'\Gamma'\Lambda+\Lambda'\Lambda\end{array}\right](-\gamma)g^{\gamma}G^{1-\gamma}=0.$$

By simplifying,

$$g_t+\left[\mu_w'+\frac{1-\gamma}{\gamma}M'(\Gamma'\Gamma)^{-1}\Gamma'\Omega\right]g_w+\frac{1}{2}tr(\Omega'\Omega g_{ww})$$
$$+\left[\frac{1-\gamma}{\gamma}u-\frac{1-\gamma}{\gamma}M'(\Gamma'\Gamma)^{-1}\Gamma'\Lambda-\frac{1-\gamma}{2(-\gamma)^2}M'(\Gamma'\Gamma)^{-1}M\right]g=0. \tag{A-2}$$